MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(An Autonomous Institution – UGC, Govt. of India)

Recognizes under 2(f) and 12(B) of UGC ACT 1956

(Affiliated to JNTUH, Hyderabad, Approved by AICTE –Accredited by NBA & NAAC-"A" Grade-ISO 9001:2015 Certified)

MATHEMATICS - II

B.Tech – I Year – II Semester

DEPARTMENT OF HUMANITIES AND SCIENCES

INDEX

I Year B.Tech -II SEM

(R17A0022)MATHEMATICS-II

Objectives

- The objective of interpolation is to find an unknown function which approximates the given data points and the objective of curve fitting is to find the relation between the variables x and y from given data and such relationships which exactly pass through the data (or) approximately satisfy the data under the condition of sum of least squares of errors.
- The aim of numerical methods is to provide systematic methods for solving problems in a numerical form using the given initial data and also used to find the roots of an equation and to solve differential equations.
- In the diverse fields like electrical circuits, electronic communication, mechanical vibration and structural engineering, periodic functions naturally occur and hence their properties are very required. Indeed, any periodic and non periodic function can be best analyzed in one way by Fourier series method.
- PDE aims at forming a function with many variables and also their solution methods .Method of separation of variables technique is learnt to solve typical second order PDE.
- Properties of Laplace Transform, Inverse Laplace Transform and Convolution theorem

UNIT – I: Solution of Algebraic, Transcendental Equations and Interpolation

Solution of Algebraic and Transcendental Equations: Introduction – Graphical interpretation of solution of equations .The Bisection Method – Regula-Falsi Method – The Iteration Method – Newton-Raphson Method.
Interpolation: Introduction-Errors in polynomial interpolation-Finite differences- Forward Differences- Backward differences –Central differences – Symbolic relations and separation of symbols-Differences of a polynomial-Newton's formulae for interpolation – Central difference interpolation Formulae – Gauss Central Difference Formulae – Interpolation with unevenly spaced points-Lagrange's Interpolation formula.

UNIT – II : Numerical Techniques

Numerical integration: Generalized Quadrature-Trapezoidal rule, Simpson's $1/3^{rd}$ and $3/8^{th}$ Rule.
Numerical solution of Ordinary Differential equations: Solution by Taylor's series method –Picard's Method of successive Approximation- single step methods-Euler's Method-Euler's modified method, Runge-Kutta Methods.
Curve fitting: Fitting a straight line –Second degree curve-exponential curve-power curve by method of least squares.

UNIT – III: Fourier series
Definition of periodic function. Fourier expansion of periodic functions in a given interval of length 2π. Determination of Fourier coefficients – Fourier series of even and odd functions – Half-range Fourier sine and cosine expansions-Fourier series in an arbitrary interval .

UNIT-IV: Partial differential equations

Introduction -Formation of partial differential equation by elimination of arbitrary constants and arbitrary functions, solutions of first order linear (Lagrange) equation and non-linear equations (Charpit's method), Method of separation of variables for second order equations and Applications of PDE to one dimensional (Heat equation).

UNIT – V Laplace Transforms and Applications

Definition of Laplace transform- Domain of the function and Kernel for the Laplace transforms- Existence of Laplace transform- Laplace transform of standard functions- first shifting Theorem,-Laplace transform of functions when they are multiplied or divided by "t"- Laplace transforms of derivatives and integrals of functions – Unit step function – second shifting theorem – Dirac's delta function- Periodic function – Inverse Laplace transform by Partial fractions-Inverse Laplace transforms of functions when they are multiplied or divided by "s", Inverse Laplace Transforms of derivatives and integrals of functions- Convolution theorem –Solving ordinary differential equations by Laplace transforms.

PRESCRIBED TEXT BOOKS:

1. Mathematics-II by Tata Mc Graw Hill Publishers.

REFERENCES:

1. Mathematical Methods by T.K.V Iyenger ,B.Krishna Gandhi and Others ,S Chand.

2. Introductory Methods by Numerical Analysis by S.S. Sastry, PHI Learning Pvt. Ltd.

3.Advanced Engineering Mathematics by Kreyszig, John Wiley & Sons Publishers.

Outcomes:

- From a given discrete data, one will be able to predict the value of the data at an intermediate point and by curve fitting, one can find the most appropriate formula for a guesses relation of the data variables. This method of analysis data helps engineers to understand the system for better interpretation and decision making.
- The student will be able to find a root of a given equation and will be able to find a numerical solution for a given differential equation. Helps in describing the system by an ODE, if possible. Also, suggests to find the solution as a first approximation.
- One will be able to find the expansion of a given function by Fourier series.
- One will be able to find a corresponding Partial Differential Equation for an unknown function with many independent variables and to find their solution.
- The student is able to solve certain differential equations using Laplace Transform. Also able to transform functions on time domain to frequency domain using Laplace transforms

UNIT-I
SOLUTION OF ALGEBRAIC, TRANSCENDENTAL EQUATIONS AND INTERPOLATION

INTRODUCTION

Using mathematical modeling, most of the problems in engg and physical and economical sciences can be formulated in terms of systems of linear or non linear equations, ordinary or partial differential equations or Integra equations. In majority of the cases, the solutions to these problems in analytical form are non-existent or difficult or not amenable for direct interpretation. In all such problems, numerical analysis provides approximate solutions practical and amenable for analysis. Numerical analysis does not strive for exaxtness.instaed.it yields approximations with specified degree of accuracy. The early disadvantages of the several numbers of computations involved has been removed through high speed computation using computers, giving results which are accurate, reliable and fast. Numerical is not only a science but also an 'art' because the choice of 'appropriate' procedure which 'best' suits to a given problem yields 'good' solutions.

Approximations curve is the graph of data obtained through measurement ofr observation. Curve fitting is the process of finding the "best fit" curve since different approximation curves can be obtained for the same data. Least squares method is the best curve fitting by a sum of exponentials, linear weighted and non-linear weighted least squares approximation.

Solution of algebraic and Transcendental equations

Introduction:

Polynomial function: A function $f(x)$ is said to be a polynomial function, if $f(x)$ is a polynomial in x. ie, $f(x) = a_0x^n + a_1x^{n-1} + \cdots \ldots \ldots \ldots .. + a_{n-1}x + a_n$

where $a_0 \neq 0$, the co-efficients $a_0, a_1 \ldots\ldots\ldots\ldots a_n$ are real constants and n is a non-negative integer.

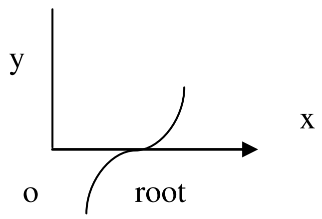
Algebraic function: A function which is a sum (or) difference (or) product of two polynomials is called an algebraic function. Otherwise, the function is called a transcendental (or) non-algebraic function.

Eg: $f(x) = x^3 - 4x^2 + 5x - 2$ is a algebraic equation

Eg: $f(x) = xcosx - e^x = 0$ is a Transcndental equation

Root of an equation: A number α is called a root of an equation $f(x) = 0$ if $f(\alpha) = 0$. We also say that α is a zero of the function.

Graphical view of a root of an equation.



The roots of an equation are the points where the graph $y = f(x)$ cuts the x-axis.

Methods to find the roots of an equation $f(x) = 0$

1.Direct methods   2.Iterative methods

1.Direct methods : We know the solution of the polynomial equations such as linear equation $ax + b = 0$, and quadratic equation $ax^2 + bx + c = 0$,using direct methods or analytical methods. Analytical methods for the solution of cubic and quadratic equations are also available.But we are unable to find roots of higher order (from fourth order) algebraic equations,and also transcendental euations.So,we go for Numerical methods i.e Iterative methods

2.Iterative methods: The following are some iteravative methods to find an approximate root of an equation

(1) **Bisection Method**
(2) **Regula- Falsi Method**
(3) **Iteration method**
(4) **Newton Raphson method**

Intermediate value theorem: If f is a real-valued continuous function on the interval $[a, b]$, and u is a number between $f(a)$ and $f(b)$, then there is a $c \in [a, b]$ such that $f(c) = u$.

Bisection method or Half-interval method:

Bisection method is a simple iteration method to find an approximate root of an equation. Suppose that given equation of the form is $f(x) = 0$.
In this method first we choose two points x_0, x_1 such that $f(x_0)$ and $f(x_1)$ will have opposite signs (i.e $f(x_0).f(x_1) < 0$) then the root lies in interval (x_0, x_1). Now we bisect this interval at x_2 ,if $f(x_2) = 0$ then x_2 is a root of an eqution otherwise the root lies in (x_0, x_2) or(x_2, x_1) accordingly $f(x_0).f(x_2) < 0$ and $f(x_2).f(x_1) < 0$.
Assume that $f(x_0).f(x_2) < 0$ then the root lies in interval (x_0, x_2) , now we bisect this interval at x_3, ,if $f(x_3) = 0$ then x_3 is a root of an eqution otherwise the root lies in (x_0, x_3) or(x_3, x_2) accordingly $f(x_0).f(x_3) < 0$ and $f(x_3).f(x_2) < 0$.
We continue this procedure till the root is found to the desired accuracy.

PROBLEMS

1. Using bisection method, find the negative root of $x^3-4x+9=0$

SOL:

Givenf(x)= x^3-4x+9

f(-1)=-1+4+9=12>0

f(-2)=-8+8+9=9>0

f(-3)=-27+12+9=-6<0

Since f(-2)>0 and f(-3)<0 therefore root lies in interval (-2,-3)= (x_0, x_1)

Bisect this interval to get next approximation x_2

i.e $x_2=\frac{-2-3}{2}$= -2.5 , $f(-2.5)>0$

Since f(-2)>0 f(-2.5)>0 f(-3)<0 therefore root lies in (-2.5,-3)

Bisect this interval to get next approximation x_3

i.e $x_3=\frac{-2.5-3}{2}$=-2.75 , $f(-2.75)<0$

Since f(-2.5)>0 f(-2.75)<0 f(-3)<0 therefore root lies in (-2.5,-2.75)

Bisect this interval to get next approximation x_4

$i.e\ x_4=\frac{-2.5-2.75}{2}$=--2.625, $f(-2.625)<0$

Since f(-2.5)>0 f(-2.625)>0 f(-2.75)<0 therefore root lies in (-2.625,-2.75)

Bisect this interval to get next approximation x_5

i.e $x_5=\frac{-2.625-2.75}{2}$=--2.6875 , $f(-2.6875)<0$

Since f(-2.625)>0 f(-2.6875)>0 f(-2.75)<0 therefore root lies in (-2.6875,-2.75)

Bisect this interval to get next approximation x_6

i.e $x_6=\frac{-2.6875-2.75}{2}$=--2.7187, $f(-2.7187)<0$

We continue this procedure till the root is found to the desired accuracy. (stop the procedure when two successive approximations are same up to four decamal places)

2). Find a root of the equation $x^3-x-1=0$ **using the bisection method in 5 – stages**

Sol Given $f(x)=x^3-x-1$

$f(1)=-1<0$

$f(2)=5>0$

∴One root lies between 1 and 2

Now see f(1) is near to 0 than f(2). So root is near to 1

so again find f(1.1),f(1.2)……

Till one is + ve and another – ve.
Clearly f(1.1)<0,f(1.2)<0
f(1.3)=-0.103<0
f(1.4)=0.344>0
Since f(1.3)<0 and f(1.4)>0 therefore root lies in interval (1.3,1.4)= (x_0, x_1)
Bisect this interval to get next approximation x_2

i.e $x_2 = \frac{1}{2}(1.3+1.4) = 1.35$

$here\ f(2) = 5 > 0$

Since f(1.3)<0 f(1.35) > 0 f(1.4)>0 therefore root lies in (1.3,1.35)
Bisect this interval to get next approximation x_3
i.e x_3=$\frac{1.3+1.35}{2}$=1.325 , $f(1.325) = 0.0012 > 0$

Since f(1.3)<0 f(1.325)>0 f(1.35)>0 therefore root lies in (1.3,1.325)
Continuing like above upto two iterations nearly same upto three decimals,we get

$\therefore Approximate\ Root = 1.32$

3) Find a root of an equation 3x= e^xusing bisection method.

Sol

Let $f(x) = 3x - e^x$

$f(1) = 0.281718 > 0$

$f(2) = -1.389056 < 0$

Since f(1)>0 and f(2)<0 therefore root lies in interval (1,2)= (x_0, x_1)
Bisect this interval to get next approximation x_2

i.e $x_2 = \frac{x_0 + x_1}{2} = 1.5$ $f(1.5) > 0$

Since f(1)>0 f(1.5)>0 f(2)<0 therefore root lies in (1.5,2)
Bisect this interval to get next approximation x_3
i.e $x_3 = \frac{1.5+2}{2}$=1.75 $f(x_3) = f(1.75) < 0$

Since f(1.5)>0 f(1.75)<0 f(2)<0 therefore root lies in (1.5,1.75)
Bisect this interval to get next approximation x_4
$i.e\ x_4$=$\frac{1.5+1.75}{2}$=1.625, $f(1.625) = 1.666 > 0$
Continuing like above up to 12 iterations we get

$$x_{11} = 1.512323$$
$$and$$
$$x_{12} = 1.512208$$

Therfore we got two successive iterations same up to three decimal places
$\therefore Approximate\ Root = 1.512$

4. Find a root of an equation x $\log_{10} x$= 1.2 using bisection method which lies between 2 and 3

Sol:

Given $f(x) = x\ log_{10} x - 1.2$

f(1)=-1.2<0

f(2)=-0.59<0

f(3)=0.23>0

Since f(2)>0 and f(3)<0 therefore root lies in interval (2,3)= (x_0, x_1)

Bisect this interval to get next approximation x_2

i.e $x_2=\frac{2+3}{2}$ =2.5

$Here\ f(2.5) < 0$

Since f(2)<0 f(2.5)<0 f(3)>0 therefore root lies in (2.5,3)

Bisect this interval to get next approximation x_3

i.e$x_3=\frac{2.5+3}{2}$=2.75 $Here\ f(x_3) = f(2.75) > 0$

Continuing like above ,we get $x_9 = 2.7453$ x_{10}=2.7406

$\therefore Approximate\ root = 2.74$

5.Find a root of an equation x = cosx using bisection method.

SOL:

$Given\ fx) = x - cos(x)$

$f(0) = 0 - cos0 = -1 < 0$

$f(1) = 1 - cos1 = 0.4597 > 0$

$then\ one\ root\ must\ be\ lies\ between\ in\ (0\ , 1)$

Here $f(1) value\ is\ near\ to\ zero\ so$

$f(0.9) = 0.2784 > 0$

$f(0.8) = 0.1033 > 0$

$f(0.7) = -0.0648 < 0$

Since f(0.7)<0 and f(8)>0 therefore root lies in interval (0.7,0.8)= (x_0, x_1)

Bisect this interval to get next approximation x_2

i.e , $x_2 = \frac{x_0+x_1}{2} = \frac{0.7+0.8}{2} = 0.75$ $f(0.75) = 0.0183 > 0$

Since f(0.7)<0 f(0.75)>0 f(0.8)>0 therefore root lies in (0.7,0.75)

Bisect this interval to get next approximation x_3

$$i.e\ x_3 = \frac{x_2 + x_0}{2} = \frac{0.7 + 0.75}{2} = 0.725 \qquad f(0.725) = -0.0235 < 0$$

Since f(0.7)<0 $f(0.725) < 0$ f(0.75)>0 therefore root lies in (0.725,0.75)

Bisect this interval to get next approximationx_4

$$i.e\ x_4 = \frac{x_2 + x_3}{2} = \frac{0.725 + 0.75}{2} = 0.7375 \qquad f(0.7375) = -0.0027 < 0$$

Since $f(0.725) < 0$ $f(0.7375) < 0$ f(0.75)>0 therefore root lies in (0.7375,0.75)

Bisect this interval to get next approximation x_5

$$i.e\ x_5 = \frac{x_2 + x_4}{2} = \frac{0.7375 + 0.75}{2} = 0.7425 \qquad f(0.7425) = 0.0057 > 0$$

We continue this procedure till the root is found to the desired accuracy. (stop the procedure when two successive approximations are same up to four decamal places)

$The\ required\ approximate\ root = 0.7392.$

6. **Find a root of an equation $3x = cosx + 1$ using bisection method.**

SOL:Given $f(x) = 3x - \cos x - 1$

$$f(0) = -2 < 0$$
$$f(1) = 1.4597 > 0$$
$$f(0.5) = -0.3776 < 0$$

Since f(0.5)<0 and f(1)>0 therefore root lies in interval (0.5,1)= (x_0, x_1)

Bisect this interval to get next approximation x_2

i.e $x_2 = \frac{x_0 + x_1}{2} = \frac{0.5+1}{2} = 0.75 \qquad f(0.75) = 0.5183 > 0$

Since f(0.5)<0 f(0.75)>0 f(1)>0 therefore root lies in (0.5,0.75)

Bisect this interval to get next approximation x_3

$i.e\ x_3 = \frac{x_2 + x_0}{2} = \frac{0.5+0.75}{2} = 0.625 \qquad f(0.625) = 0.06403 > 0$

Since f(0.5)<0 $f(0.625) > 0$ f(0.75)>0 therefore root lies in (0.5,0.625)

Bisect this interval to get next approximation x_4

$i.e\ x_4 = \frac{x_0 + x_3}{2} = \frac{0.5+0.625}{2} = 0.5625 \qquad f(0.5625) = -0.1584 < 0$

Since f(0.5)<0 $f(0.5625) < 0$ $f(0.625) > 0$ therefore root lies in (0.5625,0.625)

Bisect this interval to get next approximation x_5 $i.e\ x_5 = \frac{x_3 + x_4}{2} = \frac{0.5625+0.625}{2} =$ $0.59375 \qquad f(0.59375) = -0.0475 < 0$

We continue this procedure till the root is found to the desired accuracy.

(stop the procedure when two successive approximations are same up to four decamal places)

$So\ the\ required\ approximate\ root = 0.6074.$

7.Find the real root of the equation x^3-5x+1=0 by bisection method.

Sol: given that f(x)= x^3-5x+1

f(0)=1 >0 ,

f(1)= -3 <0

Hence the root lies between 0 and 1

Let the initial approximation be $x_0 = \frac{0+1}{2}$ =0.5

f(0.5)= -1.375 < 0

since f(0) > 0 and f(0.5)<0

therefore the root lies between 0 and 0.5

The second approximation $x_1 = \frac{0+0.5}{2}$ =0.25

f(0.25)= -0.234 < 0

since f(0) > 0 f(0.25) < 0 f(0.5)<0

therefore the root lies between 0 and 0.25

the third approximation $x_2 = \frac{0+0.25}{2}$ =0.125

Now f(0.125)=0.3749 > 0

f(0) > 0 f(0.125)>0 f(0.25) < 0

therefore the root lies between 0 and 0.125

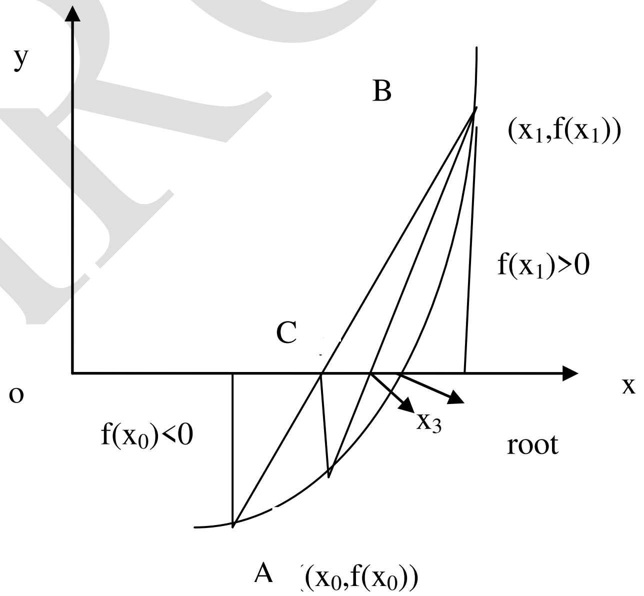
continue this procedure till the desired occurucy is obtained.

False Position Method (Regula – Falsi Method)

Using False position method we find the approximate root of the given equation $f(x)=0$ in in this method first we choose two initial approximate values x_0 *and* x_1 such that $f(x_0)$ and $f(x_1)$ will have opposite signs i.e $f(x_0).f(x_1)$ <0. Therefore the root lies in interaval (x_0, x_0)

Here two cases occur (i) $f(x_0)$<0, $f(x_1)$>0 (ii) $f(x_0)$>0, $f(x_1)$<0

FIGURE OF CASE (I)



Let $A=(x_0, f(x_0))$ *and* $B=(x_1, f(x_1))$ be the points on the curve $y=f(x)$ Then the equation to the chord AB is $\frac{y-f(x_0)}{x-x_0} = \frac{f(x_1)-f(x_0)}{x_1-x_0}$ $------(1)$

At the point C where the line AB crosses the x – axis, where $f(x)=0$ $ie,$ $y=0$

substitute $y = 0$ in equation (1), then we get

$$x = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \rightarrow (2)$$

x is given by (2) serves as an approximated value of the root, when the interval in which it lies is small. If the new value of x is taken as x_2 then (2) becomes

$$x_2 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_0)$$

$$= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \quad \text{-------------(3)}$$

Now we decide whether the root lies between

x_0 and x_2 (or) x_2 and x_1

In the above graph clearly $f(x_2)<0$

Therefore root lies between x_1 and x_2

We name that interval as (x_1, x_2)

The next approximation is given by $x_3 = \dfrac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

This will in general, be nearest to the exact root. We continue this procedure till the root is found to the desired accuracy.

The iteration process based on (3) is known as the method of false position

The successive intervals where the root lies, in the above procedure are named as

$(x_0, x_1), (x_1, x_2), (x_2, x_3)$ etc

Where $x_i < x_{i+1}$ and $f(x_0), f(x_{i+1})$ are of opposite signs.

Also $x_{i+1} = \dfrac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$

CASE(II) $f(x_0)>0$, $f(x_1)<0$

Repeate same procedure as case(i).

PROBLEMS:

1. Find an approximate root of the equation $f(x) = logx - cosx$ by using Regula-Falsi method.

Sol : Given equation is $f(x) = logx - cosx$

$f(1) = log1 - cos1 = -0.5403 < 0$

$f(2) = log2 - cos2 = 1.1093 > 0$

Since $f(1) < 0$ and $f(2) > 0$ Therefore the root lies in interval $(1,2) = (x_0, x_1)$

Since $f(x_0) = -0.5403 < 0$ and $f(x_1) = 1.1093 > 0$

The next approximation to the root is given by

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = 1.3275$$

$f(x_2) = f(1.3275) = 0.04239 > 0$

Since $f(x_0) = -0.5403 < 0$, $f(x_2) = 0.04239 > 0, f(x_1) = 1.1093 > 0$

Therefore the root lies in interval $(x_0, x_2) = (1,1.3275)$

The next approximation is

$$x_3 = \frac{x_0 f(x_2) - x_2 f(x_0)}{f(x_2) - f(x_0)} = 1.3035$$

Continue the procedure until the successive approximations are same up to four decimal places

2. Find an approximate root of the equation $f(x) = e^x sinx - 1 = 0$ by using Regula-Falsi method.

Sol: Given equation is $\boldsymbol{f(x) = e^x sinx - 1 = 0}$

$$f(0) = -1 < 0$$
$$f(1) = 1.2873 > 0$$

Since $f(0) < 0$ and $f(1) > 0$

Therefore the root lies in interval $(0,1) = (x_0, x_1)$

$f(x_0) = -1 < 0$ and $f(x_1) = 1.2873 > 0$

The next approximation to the root is given by

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = 0.4372$$

$f(x_2) = f(0.4372) = -0.3444 < 0$

$f(x_1) = 1.2873 > 0$ and $f(x_2) = -0.3444 < 0$

Therefore the root lies in interval $(0.4372,1) = (x_1, x_2)$

The next approximation is

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = 0.556$$

Continue the procedure until the successive approximations are same up to four decimal places

3. Find an approximate root of the equation $f(x) = 2x - \log_{10} x - 7 = 0$ by using Regula-Falsi method.

Sol: Given equation is $\boldsymbol{f(x) = 2x - log_{10}^x - 7 = 0}$

$$f(1) = -5 < 0$$
$$f(2) = -3.3010 < 0$$
$$f(3) = -1.4771 < 0$$
$$f(4) = 0.3979 > 0$$

Since $f(3) < 0$ and $f(4) > 0$

Therefore the root lies in interval $(3,4) = (x_0, x_1)$

$f(x_0) = -1.4771 < 0$ and $f(x_1) = 0.3979 > 0$

The next approximation to the root is given by

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = 3.7878$$

$f(x_2) = -0.0028 < 0$

$f(x_1) = 0.3979 > 0$ and $f(x_2) = -0.0228 < 0$

Therefore the root lies in interval $(3.7878,4) = (x_2, x_1)$

The next approximation is

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)} = 3.7893$$

Continue the procedure until the successive approximations are same up to four decimal places

4. **Find a root of an equation 3x= e^x using False position method**.

Sol. Let $f(x) = 3x - e^x$

Then $f(0) = -1, f(0.1) = -0.8, \ldots$
$f(0.6) = -0.0221192 < 0, f(0.7) = 0.086247 > 0$

Since $f(0.6).\ f(0.7) < 0$ and these values are near to **zero**

Therefore the root lies in the interval (0.6,0.7)= (x_0, x_1)

By False position method

The next approximation to the root is given by

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = 3.7878$$

$$= \frac{0.6 f(0.7) - 0.7 f(0.6)}{f(0.7) - f(0.6)}$$

=0.620451

Since $f(x_0)$<0 $f(x_2)$ = f(0.620451)=0.001587>0 $f(x_1)$>0

Therefore the root lies in the interval (0.6, 0.620451)= (x_0, x_2)

The next approximation to the root is given by

$$x_3 = \frac{x_0 f(x_2) - x_2 f(x_0)}{f(x_2) - f(x_0)}$$

$$= \frac{0.6 f(0.620451) - 0.620451 f(0.6)}{f(0.620451) - f(0.6)}$$

=0.619083

f (0.619083)=0.000025>0

$\therefore$ *The Approximate root is* 0.6190

5.Find the root of $x\log_{10} x - 1.2 = 0$ using Regula falsi method.

Sol:

$f(x) = x\log_{10} x - 1.2$

Here

f(2)=-0.59<0,

f(3)=0.23>0

Since f(2)<0 and f(3) >0 the root lies in the interval (2,3) = (x_0, x_1)

The next approximation to the root is given by

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{2f(3) - 3f(2)}{f(3) - f(2)}$$

=2.7195

Since $f(x_0)$<0 $f(x_2)$ = f(2.7195)=-0.0184<0 $f(x_1)$>0

Therefore the root lies in the interval(2.7195,3)= (x_2, x_1)

The next approximation to the root is given by

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$$= \frac{2.7195\, f(3) - 3f(2.7195)}{f(3) - f(2.7195)}$$

=2.7403

f(2.7403)=-0.000302<0

Clearly f(2.7403) is nearly equal to zero up to 3 decimal places

$\therefore$ *The Approximate Root is* 2.740

6.By using Regula - Falsi method, find an approximate root of the equation $x^4 - x - 10 = 0$ that lies between 1.8 and 2. Carry out three approximations

Sol.

Let us take $f(x) = x^4 - x - 10$ and $x_0 = 1.8, x_1 = 2$

Then $f(x_0) = f(1.8) = -1.3 < 0$ and $f(x_1) = f(2) = 4 > 0$

Since $f(x_0)$ and $f(x_1)$ are of opposite signs,the equation $f(x) = 0$ has a root between x_0 *and* x_1

The first order approximation of this root is

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 1.8 - \frac{2 - 1.8}{4 + 1.3} \times (-1.3)$$

$$= 1.849$$

We find that $f(x_2) = -0.161$ so that $f(x_2)$ *and* $f(x_1)$ are of opposite signs. Hence the root lies between x_2 *and* x_1 and the second order approximation of the root is

$$x_3 = x_2 - \left[\frac{x_1 - x_2}{f(x_1) - f(x_2)}\right].f(x_2)$$

$$= 1.8490 - \left[\frac{2 - 1.849}{0.159}\right] \times (-0.159)$$

$$= 1.8548$$

We find that $f(x_3) = f(1.8548)$

$$= -0.019$$

So that $f(x_3)$ *and* $f(x_2)$ are of the same sign. Hence, the root does not lie between x_2 *and* x_3.But $f(x_3)$ *and* $f(x_1)$ are of opposite signs. So the root lies between x_3 *and* x_1 and the third order approximate value of the root is

$$x_4 = x_3 - \left[\frac{x_1 - x_3}{f(x_1) - f(x_3)}\right] f(x_3)$$

$$= 1.8548 - \frac{2 - 1.8548}{4 + 0.019} \times (-0.019)$$

The approximate value of x = 1.8557

ITERATION METHOD:

Consider an equation $f(x) = 0$ which can taken in the form $x = \phi(x) \to (1)$

where $\phi(x)$ satisfies the following conditions

(i) For two real numbers a and b, $a \le x \le b$ implies $a \le \phi(x) \le b$ and

(ii) For all x^1, x^{11} lying in the interval(a, b), we have $\left|\phi(x^1) - \phi(x^{11})\right| \le m\left|x^1 - x^{11}\right|$

where m is a constant such that $0 \le m \le 1$

Then, it can be proved that the equation (1) has a unique root 'α' in the interval (a,b). To find the approximate value of this root, we start with an initial approximation x_0 of the root 'α' and find $\phi(x_0)$

We put $x_1 = \phi(x_0)$ and take x_1 as the first approximation of 'α'

Next, we put $x_2 = \phi(x_1)$ and take x_2 as the second approximation of α.Continuing the process, we get the third approximation x_3, the forth approximation x_4, and so on.

The nth approximation is given by $x_n = \phi(x_n - 1), n \ge 1 \to (2)$

In this process of finding successive approximation of the root α, an approximation of α is obtained by substituting the preceding approximation in the function $\phi(x)$ which is known.

Such a process is called on iteration process. The successive approximations $x_1, x_2 \ldots$ obtained by iteration are called the iterates. The n^{th} approximation x_n is called the n^{th} iterate.

A formula $x_n = \phi(x_n - 1), n \ge 1$ is called an iterative formula.

Convergence of An Iteration:-

Since $x_1, x_2, x_3........x_n$ are taken as successive approximations of the root α, each approximation is nearest to α than the preceding approximation, so that for large n, x_n may be taken to be almost equal to α. In other words, if the sequence $\{x_{II}\}$ converges to α, we can say that the iteration process is convergent. We state below a theorem with out proof giving a sufficient condition for the convergence of the iteration given by $x_n = \phi(x_n - 1), n \geq 1$

Note:- Let I be an interval containing a root α of the equation (1). If $[\phi^1(x)] < 1$ for all x in I, then for any value of x_0 in I, the iteration given by (2) is convergent.

PROBLEMS:

1 . **Explain the iterative method approach in solving the problems.**

Sol: In Latin the word iterate means to repeat. Iterative methods use a process of obtaining better and better estimates of solution with each iteration (or) repetitive computation. This process continues until an acceptable solution is found

The steps involved in an iterative method are

1. Develop an algorithm to solve a problem step-by-step
2. Make an initial guess or estimate for the variables (or) variables of the solution. The initial estimates should be reasonable. Success in the solution is dependent of the selection of proper initial values of variables
3. Better and better estimates are obtained in the progressive iterations by using the algorithm developed.
4. Stop the iteration process after reaching an acceptable solution, based on a reasonable criteria being met.

2. **Explain the classification of iterative method based on the number of guesses**

Sol. Iterative methods can be classified into two categories based on the number of guesses

1. **Interpolation methods – also called as bracketing methods**
2. **Extrapolation methods – also called as open end methods**

Two estimates are made for the root in the interpolation methods. One is positive value for the function $f(x)$ and the other gives a negative value for the function $f(x)$. This means that the root is bracketed between these two values

By proper selection, the gap between the two estimates can be reduced further to arrive at every small gap. Two popular methods of this type are

a) Bi-section method b)False position method

A single value, which is called as initial estimate is chosen in the extrapolation methods. The new value of the root is successively computed in each step, depending on the algorithm. This process is continued untill the difference between the values of two successive iterations is small enough to stop the iteration process. Some methods of this type are Newton- Raphson method

3.Find a root of an equation $2x - \log x = 7$ using iterative method.

SOL:

Given $f(x) = 2x - \log x - 7$

$f(1)$ =-5<0

$f(2)$ =-3.3068<0

$f(3)$ =-2.099<0

$f(4)$ = -0.387<0

$f(5)$ = 1.3905>0

Since $f(4) < 0$ and $f(5) > 0$ therefore root lies in interval (4,5)

Given equation is $2x - \log x - 7 = 0$ ---------(1)

Solve eqution (1) for x

i.e $x = \frac{\log x+7}{2}$ and $x = e^{2x-7}$ -----------------(2)

we have $x = \emptyset(x)$---------------------------(3)

comparing (2) and (3) then we have $\emptyset(x) = \frac{\log x+7}{2}$ and $\emptyset(x) = e^{2x-7}$

choose $\emptyset(x)$ such that $|\emptyset'(x)|$<1 in (4,5)

choose $\emptyset(x) = \frac{\log x+7}{2}$ for which $\emptyset'(x)=\frac{1}{2x}$ and $|\emptyset'(x)|$<1 in (4,5)

Choose $x_0 = \frac{4+5}{2}$=4.5

We know that by iteration method $x_{n+1} = \emptyset(x_n)$; n =0,1,2,3----

For $n = 0$

$x_1=\emptyset(x_0)=\emptyset(4.5) = \frac{\log(4.5)+7}{2}$=4.252

$x_2=\emptyset(x_1)=\frac{\log(4.252)+7}{2}$=4.2236

$x_3=\emptyset(x_2)=\frac{\log(4.2236)+7}{2}$=4.2203

Continue the procedure until to get a desired accuracy.

4.**By the fixed point iteration process, find the root correct to 3-decimal places, of the equation** $3x = \cos x + 1$

Sol: Given $3x = \cos x + 1$

f(0.5)= - 0.33<0

f(1)=1>0

Since $f(0.5) < 0$ and $f(1) > 0$ therefore root lies in interval (0.5,1)

The given equation is of the form $x = \phi(x)$

Where $\phi(x) = \frac{1+\cos x}{3}$

$$|\phi^1(x)| = |\sin x/3| < 1 \ for \ all \ x$$

Hence, the iteration process $x_n = \phi(x_{n-1})$ is convergent in every interval.

Choose f(0.5)= - 0.33<0 f(1)=1>0

Here f(0.5) is near to **zero** so choose x_0=0.5

Then, by iteration formula $x_n = \phi(x_{n-1})$

$$x_1 = \phi(x_0) = (\cos(0.5)+1)/3 = 0.625861$$
$$x_2 = \phi(x_1) = (\cos(0.5)+1)/3 = 0.603486$$
$$x_3 = \phi(x_2) = (\cos(0.5)+1)/3 = 0.607787$$

Similarly we get,

$$x_4 = 0.607$$

By observing these iterations, we conclude the approximate Root as **0.607** for the required root correct to three decimal places

5.Solve $x = 1 + \tan^{-1} x$ by iteration method

Sol:Let $f(x) = x - 1 - \tan^{-1} x$.

$f(0) = -1 < 0$

$f(1) = -0.785 < 0$

$f(2)$ =-0.10714<0

$f(3) = 0.7509 > 0$

Since $f(2) < 0$ and $f(3) > 0$ therefore root lies in interval (2,3)

The given equation is of the form $x = \phi(x)$

where $\phi(x) = 1 + tan^{-1}x,$

$Here\ \phi^1(x) = \frac{1}{1+x^2} < 1$

Hence the process converges in the interval(2,3)

and we take

$$x_{i+1} = 1 + \tan^{-1} x_i$$

choose $x_0 = 2$

$\therefore\ x_1 = 1 + \tan^{-1}(2) = 2.1071$

$x_2 = 1 + \tan^{-1}(2.1071) = 2.1276$

$x_3 = 1 + \tan^{-1}(2.1276) = 2.1314$

$x_4 = 1 + \tan^{-1}(2.1314) = 2.1321$

$x_5 = 1 + \tan^{-1}(2.1321) = 2.1322$

$x_6 = 1 + \tan^{-1}(2.1322) = 2.1322$

Hence the root is 2.1322(correct up to four decimal places)

NEWTON RAPHSON METHOD:

The Newton- Raphson method is a powerful and eligant method to find the root of an equation. This method is generally used to improve the results obtained by the previous methods.

Let x_0 be an approximate root of $f(x)=0$ and let $x_1 = x_0 + h$ be the correct root which implies that $f(x_1)=0$. We use Taylor's theorem and expand $f(x_1)=f(x_0+h)=0$

$$\Rightarrow f(x_0)+hf^1(x_0)=0$$

$$\Rightarrow h=-\frac{f(x_0)}{f^1(x_0)}$$

Substituting this in x_1 ,we get

$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)}$$

$\therefore x_1$ is a better approximation than x_0

Successive approximations are given by

$x_2, x_3 \dots\dots\dots x_{n+1}$ where $x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)}$

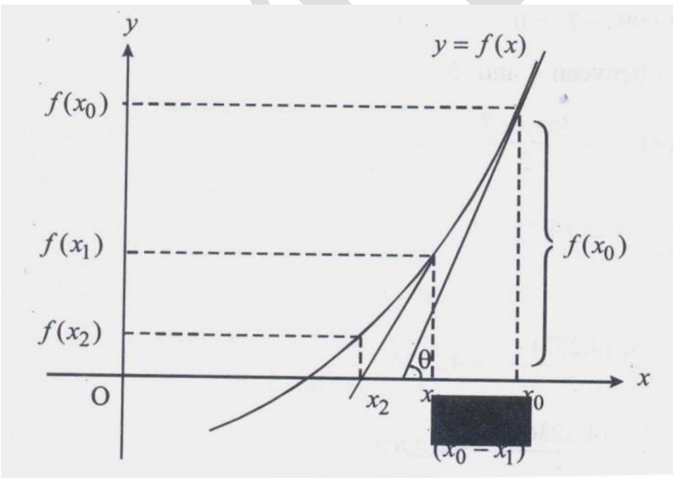
GEOMETRICAL INTERPRETATION

From below diagram $\tan\theta = \frac{opp}{adj} = \frac{f(x0)}{xo-x1}$..................(1)

But slope $=\tan\theta = f^1(x_0)$ ....(2)

From (1) and (2) we have

$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)}$$



PROBLEMS

1. Using Newton – Raphson method

a) Find square root of a number

b) Find reciprocal of a number

Sol. a) **Square root:-**

Let $f(x) = x^2 - N = 0$, where N is the number whose square root is to be found. The solution to $f(x)$ is then $x = \sqrt{N}$

Here $f'(x) = 2x$

By Newton-Raphson technique

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)} = x_i - \frac{x_i^2 - N}{2x_i}$$

$$\Rightarrow x_{i+1} = \frac{1}{2}\left[x_i + \frac{N}{x_i}\right]$$

Using the above iteration formula the square root of any number N can be found to anydesired accuracy. For example, we will find the square root of $N = 24$.

Let the initial approximation be $x_0 = 4.8$

$$x_1 = \frac{1}{2}\left(4.8 + \frac{24}{4.8}\right) = \frac{1}{2}\left(\frac{23.04+24}{4.8}\right) = \frac{47.04}{9.6} = 4.9$$

$$x_2 = \frac{1}{2}\left(4.9 + \frac{24}{4.9}\right) = \frac{1}{2}\left(\frac{24.01+24}{4.9}\right) = \frac{48.01}{9.8} = 4.898$$

$$x_3 = \frac{1}{2}\left(4.898 + \frac{24}{4.898}\right) = \frac{1}{2}\left(\frac{23.9904+24}{4.898}\right) = \frac{47.9904}{9.796} = 4.898$$

Since $x_2 = x_3$, there fore the solution to $f(x) = x^2 - 24 = 0$ is 4.898. That means, the square root of 24 is 4.898

b) Reciprocal:-

$\therefore$ The reciprocal of Let $f(x) = \frac{1}{x} - N = 0$ where N is the number whose reciprocal is to be found

The solution to $f(x)$ is then $= \frac{1}{N}$. Also, $f^1(x) = \frac{-1}{x^2}$

To find the solution for $f(x) = 0$, apply Newton – Raphson method

$$x_{i+1} = x_i - \frac{\left(\frac{1}{x_i} - N\right)}{-1/x_i^2} = x_i(2 - x_i N)$$

For example, the calculation of reciprocal of 22 is as follows

Assume the initial approximation be $x_0 = 0.045$

$$\therefore x_1 = 0.045(2 - 0.045 \times 22)$$
$$= 0.045(2 - 0.99)$$
$$= 0.0454(1.01) = 0.0454$$
$$x_2 = 0.0454(2 - 0.0454 \times 22)$$
$$= 0.0454(2 - 0.9988)$$
$$= 0.0454(1.0012) = 0.04545$$
$$x_3 = 0.04545(2 - 0.04545 \times 22)$$
$$= 0.04545(1.0001) = 0.04545$$
$$x_4 = 0.04545(2 - 0.04545 \times 22)$$
$$= 0.04545(2 - 0.99998)$$
$$= 0.04545(1.00002)$$
$$= 0.0454509$$

∴Reciprocal of 22 is 0.04545

2. Find by Newton's method, the real root of the equation $xe^x = cosx$ correct to three decimal places.

Sol. Let $cosx - xe^x = f(x)$

Then $f(0) = 1 > 0$, $f(0.5) = 0.053 > 0$ $f(0.6) = -0.267 < 0$

So root of $f(x)$ lies between 0.5 and 0.6

Here f(0.5) value is near to **zero**

f(1) is near to zero. So we take $x_0 = 0.5$ and $f^1(x) = -\sin x - (x+1)e^x$

∴ By Newton Raphson method,we have

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)} \quad \text{for i=0,1,2….}$$

First approximation is given by

$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)}$$

$$= 0.5 - \frac{0.53222}{-2.952507} = 0.68026$$

The second approximation is given by

$$x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)}$$

$$= 0.68026 - \frac{0.56569}{-3.946485}$$

$$= 0.536920$$

∴ Continue like above we have x_3= 0.51809 x_4= 0.517757

Approximate Root = 0.517

3. Find a root of an equation e^x sinx = 1 using Newton Raphson method

Sol : f(x)= e^x **sinx -1**

f(0)=-1<0

f(0.1)=-0.8<0 ….

f(0.5)=-0.209561<0

f(0.6)=0.028846>0

Since f(0.5)<0 and f(0.6)>0 the root lies in the interval(0.5,0.6)

but f(0.6) value is near to **zero**

So choose x_0=0.6

and

$f^I(x)=(\cos x+\sin x)e^x$

By applying **Newton Raphson method**,we have

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)} \quad \text{for } i=0,1,2\ldots..$$

First approximation $x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)}$

$$= 0.6 - \frac{0.028846}{2.532705} = 0.58861$$

The second approximation $x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)}$

$$= 0.588611 - \frac{0.000196}{2.498513}$$

$$= 0.588533$$

$\therefore$ *Approximate Root is* 0.588

4.Find a root of an equation x+log$_{10}$ x = 2 using Newton raphson method.

SOL:

Given $f(x) = x + log_{10} x - 2$

Here

f(1)=-1<0

f(2)=0.301>0

Since **f**(1)<0 and f(2)>0 the root lies in the interval (1,2)

Here f(2)is near to zero

So f(1.9)=0.1788>0; f(1.8)=0.0553>0

Since **f**(1.8) is near to zero

Choose $x_0 = 1.8$ then

$$f^I(x) = 1 + \frac{\log_{10} e}{x}$$

By Newton Raphson method,we have

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)} \quad \text{for } i=0,1,2\ldots..$$

$x_1 = x_0 - \frac{f^I(x_0)}{f(x_0)}$=1.8 $- \frac{0.0555}{1.2412}$ =1.7552

Now f(1.7552)=-0.00013 and $\boldsymbol{f^I(1.7552) = 1.2473}$

$x_2 = x_1 - \frac{f^I(x_1)}{f(x_1)}$ =1.7555

Now f(1.7555) =-0.00000012

Hence Approximate root is **1.7555(coorect to 4 decimal places)**

5. Using Newton – Raphson method

a) Derive formula for cube root of a number

b) Find cube root of 15.

SOL:Let f(x) = $x^3 = N$ *where N is the real number whose root to be found.*

Solution to f(x) is then $x^3 = N$ $f'(x) = 3x^2$

Newton Raphson formula to find $X_{i+1} = X_i - \frac{f(X_i)}{f'(X_i)} = X_i - \frac{X^3-N}{3X^2}$

Here f(2) = -7 < 0 and f(2.5) = 0.625 > 0

so one root lies between (2,2.5)

take initial approx value is $x_0 = 2$

using Newton Raphson formula $X_{i+1} = X_i - \frac{f(X_i)}{f'(X_i)}$

$$X_1 = 2 - \frac{(2)^3 - 15}{3(2)^2} = 2.58333$$

$$X_2 = 2.58333 - \frac{(2.58333)^3 - 15}{3(2.58333)^2} = 2.47144$$

$$X_3 = 2.47144 - \frac{(2.47144)^3 - 15}{3(2.47144)^2} = 2.46622$$

$$X_4 = 2.46622 - \frac{(2.46622)^3 - 15}{3(2.46622)^2} = 2.46621$$

$\therefore$ $x_8 \cong x_9 = 2.466221$(upto 4 decimal places) is the required approximate root.

6.Find a real root of the equation3 x = cos x + 1 Using Newton Raphson method.

f(x) = 3 x- cos x – 1

f(0) = -2 < 0

f(1) = 1.4597 > 0

$\therefore$ The root lies between 0 and 1.

Let $x_0 = 1$

using Newton Raphson formula,we have

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)} \quad \text{for i=0,1,2…..}$$

$f'(x) = 3 + \sin x$

$f'(1) = 3 + \sin 1 = 3.8414$

First approximate root $x_1 = x_0 - \frac{f^I(x_0)}{f(x_0)}$ =1-$\frac{0.4597}{3.8414}$ =0.8804

$f(0.8804) = 2.6412 - 0.6368 - 1 = 1.0044$

And $f'(0.8804) = 3.7709$

Second approximation is $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ = 1-$\frac{1.0044}{3.7709}$ =0.8804-0.2663=0.6141

$f(0.6141) = 1.8423 - 0.8172 - 1 = 0.0251$

and

$f'(0.0251) = 3.5762$

Third approximation is $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$ =0.6141- $\frac{0.0251}{3.5762}$ =0.6141-0.007=0.6071

$\therefore f(0.6071) = 1.8213 - 0.8213 - 1 = 0$

Hence Required Root is **0.6071**

7. Find the root between 0 and 1 of the equation x^3-6x+4 = 0 correct to five decimal places.

Sol: Let f(x)= x^3-6x+4

f(0)=4 >0 and f(1)= -1 <0

therefore the root lies between 0 and 1.

Let the root is nearer to 1.

So, x_0=1

$f'(x)=3x^2-6$, $f'(1)= -3$

The first approximation to the required root is

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \frac{2}{3}$ =0.66666

Second approximation is given by

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.73015$

Third approximation is given by

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.73204$

Fourth approximation is given by

$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.73205$

The root is 0.73205 correct to five decimal places

ORDER OF CONVERGENCE

The deviation from the approximate root with actual root is called **ERROR.**

Error at n th ,(n+1) iterations are

$e_n = x_n - \alpha$; $e_{n+1} = x_{n+1} - \alpha$

If $e_{n+1} \leq k\ e_n^{\,p}$ then the method is said to be of order 'p'.

NOTE:

1.The method converges very fast if 'k' is very very small and 'p' is large.

2.Regula falsi and iteration methods converge Linearly.

1.Show Bisection method converges LINEARLY.

Sol: Choose initial approximations a, b such that f(a).f(b)<0

And let first approximation be x_1

Distance between a and x_1 $=x_1$-a $=\frac{a+b}{2}$-a$=\frac{b-a}{2}$

Distance between b and x_1 $=b - x_1 = b - \frac{a+b}{2} = \frac{b-a}{2}$

Here say Root α lies between a and x_1 or b and x_1

$|x_1-\alpha| \leq \frac{b-a}{2}$

After n iterations, we get

$|x_n - \alpha| \leq \frac{b-a}{2^n}$

$|x_{n+1} - \alpha| \leq \frac{1}{2}\frac{b-a}{2^n}$

$e_{n+1} \leq \frac{1}{2} e_n^1$ $\quad \therefore$ *Bisection method converges linearly*

2.Show Newton Raphson method converges Quadratically

Sol: Let x_r be the actual root and x_i, x_{i+1} are ith,(i+1)th iterations in NRM.Then

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)}$$

$x_{i+1}\, f^1(x_i) = x_i\, f^1(x_i)$ -f(x_i)

f(x_i)= $f^1(x_i)(x_i - x_{i+1})$................(1)

Taylor's theorem around x=x_r

Is given by f(x_r)=f(x_i+h

$$= f(x_i) + (xr - xi)\ f^1(x_i) + \frac{(xr-xi)^2}{2} f^{II}(x_i) + \cdots \ldots.(2)$$

Neglecting higher order terms and sub (1) in (2) ,we get

$0 = f^1(x_i)(x_r - x_{i+1}) + (xr - xi)\ f^1(x_i) + \frac{(xr-xi)^2}{2} f^{II}(x_i)$

Solving

$$e_{i+1} = -1/2\ \left(\frac{f^{II}(x_i)}{f^I(X_i)}\right) e_i^{\,2} \quad \text{Where p=2 and k==-1/2 } \left(\frac{f^{II}(x_i)}{f^I(X_i)}\right)$$

INTERPOLATION

INTERPOLATION :

Introduction:

If we consider the statement $y = f(x); x_0 \le x \le x_n$ we understand that we can find the value of y, corresponding to every value of x in the range $x_0 \le x \le x_n$. If the function $f(x)$ is single valued and continuous and is known explicitly then the values of $f(x)$ for certain values of x like $x_0, x_1, ...x_n$ can be calculated. The problem now is if we are given the set of tabular values

x :	x_0	x_1	x_2	……………	x_n
y :	y_0	y_1	y_2	…………….	y_n

Satisfying the relation $y = f(x)$ and the explicit definition of $f(x)$ is not known, it is possible to find a simple function say $\phi(x)$ such that $f(x)$ and $\phi(x)$ agree at the set of tabulated points. This process to finding $\phi(x)$ is called interpolation. If $\phi(x)$ is a polynomial then the process is called polynomial interpolation and $\phi(x)$ is called interpolating polynomial. In our study we are concerned with polynomial interpolation

OR

Let $x_0, x_1 ----x_n$ be the values x and $y_0, y_1, y_2, ---, y_n$ be the values of y and $y = f(x)$ be a unknown function .The process to find the value of the unknown function $y = f(x)$ when the given value of x and the value of x lies within the limits $x_0 \, to \, x_n$ is called interpolation

Extrapolation:

Let $x_0, x_1 ----x_n$ be the values x and $y_0, y_1, y_2, ---, y_n$ be the values of y and y=f(x) be a unknown function .The process to find the value of the unknown function y=f(x) when the given value of x and the value of x lies outside the range of $x_0 \, to \, x_n$ is called Extrapolation

Note: If the differences of x values are equal in the given data then it is called equal spaced points otherwise it is called unequal spaced points

Note:

i) Suppose a given value of x is nearer to starting value of x then we use Newton's forward interpolation formula.

ii) Suppose a given value of x is nearer to ending value of x then we use Newton's backward interpolation formula.

iii) Suppose a given value of x is nearer to middle value of x then we use Gauss interpolation formula.

iv) Suppose the given data has unequal spaced points then we use Lagrange's interpolation formula

Finite Differences:

Finite differences play a fundamental role in the study of differential calculus, which is an essential part of numerical applied mathematics, the following are the finite differences.

1. Forward Differences 2. Backward Differences 3. Central Differences

1.Forward Differences: The Forward Difference operator is denoted by Δ , The forward differences are usually arranged in tabular columns as shown in the following table called a Forward difference table

Values of x	Values of y	First differences	Second differences	Third differences	Fourth differences
x_o	y_0				
		$\Delta y_0 = y_1 - y_0$			
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - y_0$		
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
x_3	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
x_4	y_4	$\Delta y_3 = y_4 - y_3$			

2. Backward Differences: The Backward Difference operator is denoted by ∇ and the backward difference table is

x	y	∇y	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		∇y_1			
x_1	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		∇y_3		$\nabla^3 y_4$	
x_3	y_3		$\nabla^2 y_4$		
		∇y_4			
x_4	y_4				

3.Central Difference Table: The central difference operator is denoted by δ and the central Difference table is

x	Y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
x_0	y_0				
		$\delta y_{1/2}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_4$
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{7/2}$			
x_4	y_4				

Symbolic Relations and Separation of symbols:

We will define more operators and symbols in addition to Δ, ∇ and δ already defined and establish difference formulae by Symbolic methods

Definition:- The averaging operator μ is defined by the equation $\mu y_r = \frac{1}{2}\left[y_{r+1/2} + y_{r-1/2}\right]$

Definition:- The shift operator E is defined by the equation $Ey_r = y_{r+1}$. This shows that the effect of E is to shift the functional value y_r to the next higher value y_{r+1}. A second operation with E gives $E^2 y_r = E\left(Ey_r\right) = E\left(y_{r+1}\right) = y_{r+2}$

Generalizing $E^n y^r = y_{r+n}$

Definition:-

Inverse operator E^{-1} is defined as $E^{-1}y_r = y_{r-1}$

In general $E^{-n}y_n = y_{r-n}$

Definition :-

The operator D is defined as $Dy(x) = \frac{d}{dx}[y(x)]$

Relationship Between operators:

i) Relation between Δ *and* E

Proof: We have $\Delta y_0 = y_1 - y_0$

$$= Ey_0 - y_0 = (E-1)y_0$$

$$\Rightarrow \Delta \cong E - 1 (or) E = 1 + \Delta$$

ii) $\nabla \equiv 1 - E^{-1}$

Pf: We have $\nabla y_1 = y_1 - y_0$

$$\nabla y_1 = y_1 - E^{-1}y_1$$

$$\nabla y_1 = (1 - E^{-1})y_1$$

$$\nabla \equiv 1 - E^{-1}$$

iii) $\delta \equiv E^{1/2} - E^{-1/2}$

Pf : We have $\delta y_{\frac{1}{2}} = y_1 - y_0$

$$= E^{\frac{1}{2}}y_{\frac{1}{2}} - E^{-\frac{1}{2}}y_{\frac{1}{2}}$$

$$\delta y_{\frac{1}{2}} = (E^{\frac{1}{2}} - E^{-\frac{1}{2}})y_{\frac{1}{2}}$$

$$\delta \equiv E^{1/2} - E^{-1/2}$$

iv) $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$

Pf: we have $\mu y_r = \frac{1}{2}(y_{r+\frac{1}{2}} + y_{r-\frac{1}{2}})$

$$\mu y_r = \frac{1}{2}(E^{\frac{1}{2}}y_r + E^{-\frac{1}{2}}y_r)$$

$$\mu y_r = \frac{1}{2}(E^{\frac{1}{2}} + E^{-\frac{1}{2}})y_r$$

$$\mu = \frac{1}{2}(E^{\frac{1}{2}} + E^{-\frac{1}{2}})$$

v) $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$

Pf: L.H.S = $\mu^2 = \left[\frac{1}{2}(E^{\frac{1}{2}} + E^{-\frac{1}{2}})\right]^2$

$= \frac{1}{4}(E + E^{-1} + 2)$

$= \frac{1}{4}\left[\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)^2 + 4\right]$

$= \frac{1}{4}(\delta^2 + 4)$=R.H.S

vi). Prove that $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{1}{4}\delta^2}$

Pf: Let R.H.S= $\frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{1}{4}\delta^2}$

$= \frac{1}{2}\delta[\delta + 2\sqrt{1 + \frac{1}{4}\delta^2}\ ]$

$= \frac{1}{2}\delta[\delta + \sqrt{4 + \delta^2}\ ]$

$= \frac{1}{2}\delta[(E^{\frac{1}{2}} - E^{-\frac{1}{2}}) + \sqrt{4 + (E^{\frac{1}{2}} - E^{-\frac{1}{2}})^2}$

$= \frac{1}{2}\delta[(E^{\frac{1}{2}} - E^{-\frac{1}{2}}) + \sqrt{(E^{\frac{1}{2}} + E^{-\frac{1}{2}})^2}$

$= \frac{1}{2}\delta\left[\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) + (E^{\frac{1}{2}} + E^{-\frac{1}{2}})\right.$

$= \frac{1}{2}\delta.2.E^{\frac{1}{2}}$

$= \delta.E^{\frac{1}{2}}$

$= (E^{\frac{1}{2}} - E^{-\frac{1}{2}}).E^{\frac{1}{2}}$

$= E - 1 = \Delta$ =R.H.S.

vii) Relation between the Operators D and E

Using Taylor's series we have, $y(x+h) = y(x) + hy^{1}(x) + \frac{h^2}{2!}y^{11}(x) + \frac{h^3}{3!}y^{111}(x) + ----$

This can be written in symbolic form

$$Ey_x = \left[1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + ----\right]y_x \quad = e^{hD}.y_x$$

$$E = e^{hd}$$

❖ If $f(x)$ is a polynomial of degree n and the values of x are equally spaced then $\Delta^n f(x)$ is a constant

Note:

1. As $\Delta^n f(x)$ is a constant, it follows that $\Delta^{n+1} f(x) = 0, \Delta^{n+2} f(x) = 0, ...$

2. The converse of above result is also true. That is, if $\Delta^n f(x)$ is tabulated at equal spaced intervals and is a constant, then the function $f(x)$ is a polynomial of degree n

3. $\Delta^2 f(x) = \Delta(\Delta(f(x))$

Problems :

1.Evaluate

$(i)\Delta \cos x$

$(ii)\Delta^2 \sin(px+q)$

$(iii)\Delta^n e^{ax+b}$

(iv). If the interval of difference is unity then prove that

$$\Delta[x(x+1)(x+2)(x+3) = 4(x+1(x+2)(x+3))$$

Sol: Let h be the interval of differencing

$(i)\Delta \cos x = \cos(x+h) - \cos x$

$$= -2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}$$

$(ii)\Delta \sin(px+q) = \sin[p(x+h)+q] - \sin(px+q)$

$$= 2\cos\left(px+q+\frac{ph}{2}\right)\sin\frac{ph}{2}$$

$$= 2\sin\frac{ph}{2}\sin\left(\frac{\pi}{2}+px+q+\frac{ph}{2}\right)$$

$$\Delta^2\sin(px+q) = 2\sin\frac{ph}{2}\Delta\left[\sin\left[px+q+\frac{1}{2}(\pi+ph)\right]\right]$$

$$= \left[2\sin\frac{ph}{2}\right]^2 \sin\left[px+q+\frac{1}{2}(\pi+ph)\right]$$

$(iii)\ \Delta e^{ax+b} = e^{a(x+h)+b} - e^{ax+b}$

$$= e^{(ax+b)}(e^{ah-1})$$

$$\Delta^2 e^{ax+b} = \Delta[\Delta(e^{ax+b})] = \Delta[(e^{ah}-1)(e^{ax+b})]$$

$$= (e^{ah}-1)^2\Delta(e^{ax+b})$$

$$= (e^{ah}-1)^2 e^{ax+b}$$

Proceeding on, we get $\Delta^n\left(e^{ax+b}\right) = \left(e^{ah}-1\right)^n e^{ax+b}$

iv) Let $f(x) = x(x+1)(x+2)(x+3)$

given $h = 1$

we know that $\Delta f(x) = f(x+h) - f(x)$

$$\Delta[x(x+1)(x+2)(x+3)] = (x+1)(x+2)(x+3)(x+4)) - x(x+1)(x+2)(x+3)$$
$$= (x+1)(x+2)(x+3)[x+4-x]$$
$$= \mathbf{4}(x+1)(x+2)(x+3)$$

2.Find the missing term in the following data

x	0	1	2	3	4
y	1	3	9	-	81

Why this value is not equal to 3^3 . Explain

Solution: Consider $\Delta^4 y_0 = 0$

$\Rightarrow y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$

Substitute given values, we get

$81 - 4y_3 + 54 - 12 + 1 = 0 \Rightarrow y_3 = 31$

From the given data we can conclude that the given function is $y = 3^x$. To find y_3 , we have to assume that y is a polynomial function, which is not so. Thus we are not getting $y = 3^3 = 27$

Equally Spaced : If the differences of x values are equal in the given data then it is called equal spaced points otherwise it is called unequal spaced points

Newton's Forward Interpolation Formula: Given the set of $(n+1)$ values $(x_0, y_0), (x_1, y_1), ---x_n, y_n)$ of x and y. It is required to find a polynomial of n[th] degree $y_n(x)$ such that y and $y_n(x)$ agree at the tabular points with x's equidistant (i.e.) $x_i = x_0 + ih$ (i = 0, 1, 2…..n) then the **Newton's forward interpolation formula** is given by

$$y = f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + --$$
$$+ \frac{p(p-1)(p-2)----(p-(n-1))}{n!}\Delta^n y_0$$

where $p = \frac{x - x_0}{h}$

Note : this formula is used when value of x is located near beginning of tabular values

Problems :

1.Find the melting point of the alloy containing 54% of lead, using appropriate interpolation formula

Percentage of lead(p)	50	60	70	80
Temperature $(Q^\circ c)$	205	225	248	274

Solution: The difference table is

x	y	Δ	Δ^2	Δ^3
50	205			
		20		
60	225		3	
		23		0
70	248		3	
		26		
80	274			

Let temperature = $f(x)$

We have $x = 54, x_0 = 50, h = 10 \quad p = \frac{x-x_0}{h} = 0.4$

By Newton's forward interpolation formula

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + ----$$

$$f(54) = 205 + 0.4(20) + \frac{0.4(0.4-1)}{2!}(3) + \frac{(0.4)(0.4-1)(0.4-2)}{3!}(0)$$

=205+8-0.36 =212.64. Melting point = 212.64

2. The population of a town in the decimal census was given below. Estimate the population for the 1895

Year x	**1891**	**1901**	**1911**	**1921**	**1931**
Populationin thousands	**46**	**66**	**81**	**93**	**101**

Solution: The forward difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

$$46 + (0.4)(20) + \frac{(0.4)(0.4-1)}{6} - (-5) + \frac{(0.4-1)0.4(0.4-2)}{6}(2)$$

given $h = 10, x_0 = 1891, x = 1985$ then $p = 2/5 = 0.4$

By Newton's forward interpolation formula

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p+1)}{2!}\Delta^2 y_0 + \frac{p(p+1)(p+2)}{3!}\Delta^3 y_0 + ----∴$$

$$f(1895) = + \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{24}(-3)$$

=54.45 thousands

3. Find y (1.6) using Newton's Forward difference formula from the table

x	1	1.4	1.8	2.2
y	**3.49**	**4.82**	**5.96**	**6.5**

Solution: The difference table is

X	y	Δy	Δ^2y	Δ^3y
1	3.49			
1.4	4.82	1.33	-0.81	
1.8	5.96	1.14	-0.60	-1.41
2.2	6.5	0.54		

Let = 1.6 , x_0=1, h=1.4 -1=0.4, $p = \frac{x-x_0}{h} = \frac{3}{2}$

Using Newton's forward difference formula, we have

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p+1)}{2!}\Delta^2 y_0 + \frac{p(p+1)(p+2)}{3!}\Delta^3 y_0 + ----$$

$$f\,(1.6) = 3.49 + 3/2(1.33) + \frac{\frac{3}{2}\frac{5}{2}}{2}(-0.81) + \frac{\frac{3}{2}\frac{57}{22}}{6}(-1.41)$$

= 4.9656

4.Find the cubic polynomial which takes the following values

X	0	1	2	3
Y=f(x)	**1**	**2**	**1**	**10**

Hence evaluate f(4).

Sol: The forward difference table is given by

X	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	**1**			
		1		
1	**2**		-2	
		-1		12
2	**1**		10	
		9		
3	**10**			

$P=\frac{x-0}{h}=x$; h=1

Using newton's forward interpolation formula, we get

$$Y = y_0 + \frac{x}{1}\Delta y_0 + \frac{x(x-1)}{1.2}\Delta^2 y_0 + \frac{x(x-1)(x-2)}{1.2.3}\Delta^3 y_0$$

$$= 1+x(1)+ \frac{x(x-1)}{2}(-2)+ \frac{x(x-1)(x-2)}{6}(12)$$

$$= 2x^3-7x^2+6x+1$$

Which is the required polynomial.

To compute f(4), we take $x_n=3$, x=4

So that $p=\frac{x-x_n}{h}=1$

Using Newton's backward interpolation formula , we get

$$Y_4 = y_3+p\nabla y_3+\frac{p(p+1)}{1.2}\nabla^2 y_3+\frac{p(p+1)(p+2)}{1.2.3}\nabla^3 y_3$$

$$= 10+9+10+12$$

$$= 41$$

Which is the same value as that obtained by substituting x=4 in the cubic polynomial $2x^3-7x^2+6x+1$.

Newton's Backward Interpolation Formula: Given the set of $(n+1)$ values $(x_0, y_0), (x_1, y_1), ---x_n, y_n)$ of x and y. It is required to find a polynomial of n[th] degree $y_n(x)$ such that y and $y_n(x)$ agree at the tabular points with x's equidistant (i.e.) $x_i = x_0+ih$ (i = 0, 1, 2…..n) then the **Newton's backward interpolation formula** is given by

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + --- + \frac{p(p+1)----[p+(n-1)]}{n!}\nabla^n y_0$$

Where $p=\frac{x-x_n}{h}$

Note : This formula is used when value of x is located near end of tabular values

Problems :

1. The population of a town in the decimal census was given below. Estimate the population for the 1925

Year x	1891	1901	1911	1921	1931
Population in thousands	**46**	**66**	**81**	**93**	**101**

Solution : The backward difference table is

x	y	∇	∇^2	∇^3	∇^4
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

given $h = 10, x_n = 1931, x = 1925$ then $p = \frac{x-x_n}{h} = \frac{1925-1931}{10} = -0.6$

By Newton's backward interpolation formula

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + - + \frac{p(p+1) \ldots . [p+(n-1)]}{n!}\nabla^n y_0$$

$$\therefore f(1925) = 101 + (-0.6)(8) + \frac{(-0.6)(0.4)}{2}(-4)$$

$$+ \frac{(-0.6)(0.4)(1.4)}{6}(-1) + \frac{(-0.6)(0.4)(1.4)(2.4)}{24}(-3)$$

$$= 96.21$$

2.Find $y(42)$ from the following data. Using Newton's interpolation formula

x	20	25	30	35	40	45
y	354	332	291	260	231	204

Solution: since x=42 is located near end of the tabular values therefore we use NBIF

the backward difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
20	354					
		-22				
25	332		-19			
		-41		29		
30	291		10		-37	
		-31		-8		45
35	260		2		8	
		-29		0		
40	231		2			
		-27				
45	204					

Given $x = 42$ and $x_n = 45$, $h = 5$, then $p = \frac{x-x_0}{h} = -0.6$

We know that NBIF

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla^5 y_n$$

$$y(42) = 204 + (-0.6)(-27) + \frac{(-0.6)(-0.6+1)}{2}(2)+0+\frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{24}(8) + \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)(-0.6+4)}{120}(45)$$

=234.44

Central Difference Interpolation: The middle part of the forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_{-4}	y_{-4}					
		Δy_{-4}	$\Delta^2 y_{-4}$			
x_{-3}	y_{-3}					
		Δy_{-3}	$\Delta^2 y_{-3}$	$\Delta^3 y_{-4}$	$\Delta^4 y_{-4}$	$\Delta^5 y_{-4}$
x_{-2}	y_{-2}					
		Δy_{-2}	$\Delta^2 y_{-2}$	$\Delta^3 y_{-3}$	$\Delta^4 y_{-3}$	$\Delta^5 y_{-3}$
x_{-1}	y_{-1}					
		Δy_{-1}	$\Delta^2 y_{-1}$	$\Delta^3 y_{-2}$	$\Delta^4 y_{-2}$	$\Delta^5 y_{-2}$
x_0	y_0					
		Δy_0	$\Delta^2 y_0$	$\Delta^3 y_{-1}$	$\Delta^4 y_{-1}$	$\Delta^5 y_{-1}$
x_1	y_1					
		Δy_1	$\Delta^2 y_1$	$\Delta^3 y_0$	$\Delta^4 y_0$	
x_2	y_2					
		Δy_2	$\Delta^2 y_2$	$\Delta^3 y_1$		
x_3	y_3					
		Δy_3				
x_4	y_4					

1.Gauss's forward Interpolation Formula: Given the set of $(n+1)$ values $(x_0, y_0), (x_1, y_1), ---x_n, y_n)$ of x and y. It is required to find a polynomial of n[th] degree $y_n(x)$ such that y and $y_n(x)$ agree at the tabular points with x's equidistant (i.e.) $x_i = x_0+ih$ (i = 0, 1, 2…..n) then the **Gauus Forward interpolation formula** is given by

$$\boldsymbol{y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{3!}\Delta^3 y_{-1} + \frac{p(p-1)(p+1)(p-2)}{4!}\Delta^4 y_{-2} + ---}$$

Where $p = \frac{x-x_0}{h}$

Note:- We observe from the difference table that

$\Delta y_0 = \delta y_{1/2}, \Delta^2 y_{-1} = \delta^2 y_0, \Delta^3 y_{-1} = \delta^3 y_{1/2}, \Delta^4 y_{-2} = \delta^4 y_0$ and so on. Accordingly the formula (4) can be rewritten in the notation of central diff

$$y_p = [y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 y_{1/2}$$
$$+\frac{(p+1)(p-1)p(p-2)}{4!}\delta^4 y_0 + ----]$$

2.Gauss's Backward Interpolation formula: Given the set of $(n+1)$ values $(x_0, y_0), (x_1, y_1), ---x_n, y_n)$ of x and y. It is required to find a polynomial of n^{th} degree $y_n(x)$ such that y and $y_n(x)$ agree at the tabular points with x's equidistant (i.e.) $x_i = x_0 + ih$ (i = 0, 1, 2…..n) then the **Gauus Backward interpolation formula** is given by

$$y = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-2}$$
$$+ \frac{p(p+1)(p-1)(p+2)}{4!}\Delta^4 y_{-2} + \ldots$$

Note: Gauss forward and Backward formulae used when x is located middle of the tabular values

Problems :

1.**Use Gauss Forward interpolation formula to find $f(3.3)$ from the following table**

x	1	2	3	4	5
$y = f(x)$	**15.30**	**15.10**	**15.00**	**14.50**	**14.00**

Solution: the difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1 x_{-2}	15.3 y_{-2}				
		-0.2			
2 x_{-1}	15.1 y_{-1}		0.1		
		-0.1		-0.5	
3 x_0	15.0 y_0		-0.4 $\Delta^2 y_{-1}$		0.9$\Delta^4 y_{-2}$
		-0.5 Δy_0		0.4$\Delta^3 y_{-1}$	
4 x_1	14.5 y_1		0.0		
		-0.5			
5 x_2	14.0 y_2				

Given x=3.3 , x_0=3 ,h=1 hence $p = \frac{x-x_0}{h} = 0.3$

We know that Gauss forward interpolation formula is

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+\frac{(p+1)(p-1)p(p-2)}{4!}(\Delta^4 y_{-2}) + ----] \rightarrow (4)$$

$$=15+(0.3)(0.5)+\frac{(0.3)(0.3-1)}{2}(-0.4) + \frac{(0.3)(0.09-1)}{6}(0.4)+ \frac{(0.3)(0.09-1)(0.3-2)}{24}(0.9)$$

$=14.9$

2. **Find f (2.5) using following Table**

x	**1**	**2**	**3**	**4**
y	**1**	**8**	**27**	**64**

Solution: The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	1			
		7		
2	8		12	
		19		6
3	27		18	
		37		
4	64			

h = 1

$$P = \frac{X - X_0}{h} = \frac{2.5-2}{1} = 0.5$$

Using Gauss Forward interpolation formula,

$$= 8 + (0.5)19 + \frac{(0.5)(-0.5)}{2}(12) + \frac{(0.5-1)(0.5)(1.5+1)}{6}(6)$$

$= 15.625$

3. Use Gauss forward interpolation formulae to find f(3.3) from the following

x	**1**	**2**	**3**	**4**	**5**
y	**15.30**	**15.10**	**15.00**	**14.50**	**14.00**

Solution:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	15.30				
		-0.20			
2	15.10		0.10		
		-0.10		-0.50	
3	15.00		-0.40		0.90
		-0.50		0.40	
4	14.50		0.00		
		-0.50			
5	14.00				

$$P=\frac{3.3-3}{1}=0.3$$

$$=15+(0.3)(-0.5)+\frac{(0.3)(-0.4)(-0.7)}{2}+(0.3)(0.4)\frac{(-0.7)(1.3)}{6}$$
$$+\frac{(0.3)(-0.7)(1.3)(-1.3)}{24}(-0.9)=14.8604925=14.9$$

4. Find f(2.36) from the following table

x:	1.6	1.8	2.0	2.2	2.4	2.6
y:	**4.95**	**6.05**	**7.39**	**9.03**	**11.02**	**13.46**

Solution:

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
1.6	4.95	1.1	0.24	0.06	-0.01	0.06
1.8	6.05	1.34	0.3	0.05	0.05	
2.0	7.39	1.64	0.35	0.1		
2.2 x_0	9.03 y_0	1.99	0.45			
2.4	11.02	2.44				
2.6	13.46					

here we have $x = 2.36$, $x_0 = 2.2$, $h = 0.2$, $p = \frac{x-x_0}{h} = 0.8$

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$
$$+\frac{(p+1)(p-1)p(p-2)}{4!}(\Delta^4 y_{-2}) + -----] \rightarrow (4)$$

Substituting all above values in the formula then

$f(2.36) =$

$$9.03 + (0.8)(1.99) +$$
$$\frac{(0.8)(0.8-1)}{2}(0.35)+\frac{(0.8+1)(0.8)(0.8-1)}{6}(0.1)+\frac{(0.8+1)(0.8)(0.8-1)(0.8-2)}{24}(0.05)$$

$=10.02$

5. Find f(22) from the following table using Gauss forward formula

x	20	25	30	35	40	45
y	354	332	291	260	231	204

Solution : the middle part of the difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
$20x_0$	$354y_0$					
		-22				
25	332		-19			
		-41		29		
30	291		10		-37	
		-31		-8		45
35	260		2		8	
		-29		0		
40	231		2			
		-27				
45	204					

Given $x = 22$ and $x_0 = 20$, $h = 5$, then $p = \frac{x-x_0}{h} = 0.4$

The Gauss forward formula is

$y = y_0 + p\Delta y_0$

$= 354 + (0.4)(-22)$

$= 345.2$

6. Find by Gauss's Backward interpolating formula the value of y at x=1936, using the following table.

x	1901	1911	1921	1931	1941	1951
y	12	15	20	27	39	52

Solution: The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1901 x_{-3}	12 y_{-3}					
		3				
1911 x_{-2}	15 y_{-2}		2			
		5		0		
1921 x_{-1}	20 y_{-1}		2		3	
		7 Δy_{-1}		$3\Delta^3 y_{-2}$		$-10\Delta^5 y_{-3}$
1931 x_0	27 y_0		5 $\Delta^2 y_{-1}$		$7\Delta^4 y_{-2}$	
		12		-4		
1941 x_1	39 y_1		1			
		13				
1951 x_2	52 y_2					

Given x=1936 and let x_0 =1931 and h=10 then $p = \frac{x-x_0}{h} = 0.5$

By Gauss backward interpolation formula we have

$$y = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + ---$$

$$=27+(0.5)(7)+\frac{0.5)(0.5+1)}{2}(5) + \frac{(0.5)(1.5)(-0.5)}{6}(3) + \frac{(0.5)(1.5)(-0.5)(-1.5)}{24}(-7) + \frac{(0.5)(1.5)(-0.5)(-1.5)(2.5)}{120}(-10)$$

=32.345

7. Using Gauss back ward difference formula, find y(8) from the following table

x	0	5	10	15	20	25
y	7	11	14	18	24	32

Solution: Solution: The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0 x_{-2}	7 y_{-2}					
		4				
5 x_{-1}	11 y_{-1}		-1			
		3		2		
10 x_0	14 y_0		1		-1	
		4		1		0
15 x_1	18 y_1		2		-1	
		6		0		
20 x_2	24y_2		2			
		8				
25 x_3	32 y_3					

Given x=8 and let x_0 =10 and h=5 then $p=\frac{x-x_0}{h}=-0.4$

By Gauss backward interpolation formula we have

$$y=y_0+p\Delta y_{-1}+\frac{(p+1)p}{2!}\Delta^2 y_{-1}+\frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2}$$
$$+\frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2}+-$$

$$14+(0.4)(3)+\frac{(-0.4)(-0.4+1)}{2}(1)+\frac{(-0.4)(-0.4+1)(-0.4-1)}{6}(2)+$$
$$\frac{(-0.4)(-0.4+1)(-0.4-1)(-0.4-2)}{24}(-1)=12.704$$

Lagrange's Interpolation Formula: Let $f(x)$ be continuous and differentiable (n+1) times in the interval (a,b). Given the (n+1) points as$(x_0, y_0), (x_1, y_1), (x_2, y_2)\ldots. (x_n,y_n)$ where values of x not necessarily be equally spaced then the interpolating polynomial of degree 'n' say $f(x)$ is given by

$$f(x)=\frac{(x-x_1)(x-x_2).......(x-x_n)}{(x_0-x_1)(x_0-x_2).........(x_0-x_n)}f(x_0)+\frac{(x-x_0)(x-x_2).....(x-x_n)}{(x_1-x_0)(x_1-x_2)....(x_1-x_n)}f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_2)........(x-x_n)}{(x_2-x_0)(x_2-x_1).........(x_2-x_n)}f(x_2)+\ \ldots\ldots+\frac{(x-x_0)(x-x_1)........(x-x_{n-1})}{(x_n-x_0)(x_n-x_1).........(x_n-x_{n-1})}f(x_n)$$

Note : This formula is used when values of *x* are unequally spaced and equally spaced

PROBLEMS:-

1. **Using Lagrange formula, calculate** $f(3)$ **from the following table**

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

Solution: Given $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 4, x_4 = 5, x_5 = 6$

$f(x_0)=1, f(x_1)=14, f(x_2)=15, f(x_3)=5, f(x_4)=6, f(x_5)=19$

From Lagrange's interpolation formula

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)}f(x_0)$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}f(x_1)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)}f(x_2)$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}f(x_5)$$

Here $x=3$ then

$$f(3)=\frac{(3-1)(3-2)(3-4)(3-5)(3-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)}\times 1+$$

$$\frac{(3-0)(3-2)(3-4)(3-5)(3-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)}\times 14+$$

$$\frac{(3-0)(3-1)(3-4)(3-5)(3-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)}\times 15+$$

$$\frac{(3-0)(3-1)(3-2)(3-5)(3-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)}\times 5+$$

$$\frac{(3-0)(3-1)(3-2)(3-4)(3-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)}\times 6+$$

$$\frac{(3-0)(3-1)(3-2)(3-4)(3-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)}\times 19$$

$$=\frac{12}{240}-\frac{18}{60}\times 14+\frac{36}{48}\times 15+\frac{36}{48}\times 5-\frac{18}{60}\times 6+\frac{12}{40}\times 19$$

$=0.05-4.2+11.25+3.75-1.8+0.95$

$=10$

$f(x_3)=10$

2. Find $f(3.5)$ using Lagrange method of 2^{nd} and 3^{rd} order degree polynomials.

x	1	2	3	4
$f(x)$	1	2	9	28

Sol: By Lagrange's interpolation formula For $n=4$,we have

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+$$
$$\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)+$$
$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+$$
$$\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)+$$

$$\therefore f(3.5)=\frac{(3.5-2)(3.5-3)(3.5-4)}{(1-2)(1-3)(1-4)}(1)+\frac{(3.5-1)(3.5-3)(3.5-4)}{(2-1)(2-3)(2-4)}(2)+$$
$$\frac{(3.5-1)(3.5-2)(3.5-4)}{(3-1)(3-2)(3-4)}(9)+\frac{(3.5-1)(3.5-2)(3.5-3)}{(4-1)(4-2)(4-3)}(28)$$

=0.0625+(-0.625)+8.4375+8.75

=16.625

$$Now\ \ f(x)=\frac{(x-2)(x-3)(x-4)}{-6}(1)+\frac{(x-1)(x-3)(x-4)}{2}(2)$$
$$++\frac{(x-1)(x-2)(x-4)}{(-2)}(9)+\frac{(x-1)(x-2)(x-3)}{6}(28)$$

$$=\frac{(x^2-5x+6)(x-4)}{-6}+(x^2-4x+3)(x-4)+\frac{(x^2-3x+2)(x-4)}{-2}(9)$$
$$+\frac{(x^2-3x+2)(x-3)}{6}(28)$$

$$=\frac{x^3-9x^2+26x-24}{-6}+x^3-8x^2+19x-12+\frac{x^3-7x^2+14x-8}{-2}(9)$$
$$+\frac{x^3-6x^2+11x-6}{6}(28)$$

$$=\frac{\left[-x^3+9x^2-26x+24+6x^3-48x^2+114x-72-27x^3+189x^2-378x+216+308x+28x^3-168x^2-168\right]}{6}$$

$$=\frac{6x^3-18x^2+18x}{6}\Rightarrow f(x)=x^3-3x^2+3x$$

$$\therefore f(3.5) = (3.5)^3 - 3(3.5)^2 + 3(3.5) = 16.625$$

3. Find f (4) use Lagrange's interpolation formulae.

x	0	2	3	6
Y=f(x)	-4	2	14	158

Soluiotn : $f(x) = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} y_1 + \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}$

$$Y_2 + \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} y_3 + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} Y_4$$

Where $x = 4, x_1 = 0, x_2 = 2, x_3 = 3, x_4 = 6$

$$= \frac{(4-2)(4-3)(4-6)}{(-2)(-3)(-6)} \times (-4) +$$

$$\frac{(4)(1)(-2)}{2(-1)(-4)} \times (2) + \frac{4 \times 2 \times (-2)}{3 \times 1 \times 3(-3)} \times 14$$

$$= \frac{4(2)(1)}{6(4)(3)} \times 158$$

$$= \frac{-4}{9}(-2) + \frac{224}{9} + \frac{158}{9} = \frac{-4-18+224+158}{9}$$

$$= 40$$

4.The following are the measurements T made on curverecorded by the oscilograph representing a change of current I due to a change in condn s of anelectric current

T	1.2	2	2.5	3
I	1.36	0.58	0.34	0.2

Solution :

Since data is unequispaced,we use Lagrange's interpolation

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$y = \frac{(1.6-1.2)(1.6-2)(1.6-3)}{(1.6-1.2)(1.6-2)(1.6-3)}(1.36) + \frac{(1.6-1.2)(1.6-2.5)(1.6-3)}{(2-1.2)(2-2.5)(1.6-3)}(0.58)$$

$$+ \frac{(1.6-1.2)(1.6-2)(1.6-3)}{(1.6-1.2)(1.6-2)(1.6-3)}(0.34) + \frac{(1.6-1.2)(1.6-2)(1.6-2.5)}{(1.6-1.2)(1.6-2)(1.6-2.5)}(0.2)$$

=0.8947 $\therefore I = 0.8947$

5.Find the parabola passing through points(0,1),(1,3) and(3,55) using Lagrange's Interpolation Formula.

x	0	1	3
y	1	3	55

Solution : Given Lagrange's interpolation formula is

$$y=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1$$

$$+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

$$y=\frac{(x-1)(x-3)}{(0-1)(0-3)}+\frac{(x-0)(x-3)}{(1-0)(1-3)}(3)$$

$$+\frac{(x-0)(x-1)}{(3-0)(3-1)}(55)$$

$=\frac{1}{6}[48x^2-36x+6]$

$=8x^2-6x+1$

6.A Curve passes through the points (0,18),(1,10),(3,-18) and (6,90). Find the slope of the curve at x=2.

x	0	1	3	6
y	18	10	-18	90

Solution : Given data is

Since data is unequispaced,we use Lagrange's interpolation

$$y=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$y=\frac{(x-1)(x-3)(x-6)}{(0-1)(0-3)(0-6)}18+\frac{(x-0)(x-3)(x-6)}{(1-0)(1-3)(1-6)}10$$

$$+\frac{(x)(x-1)(x-6)}{(3-0)(3-1)(3-6)}(-18)+\frac{(x)(x-1)(x-3)}{(6)(6-1)(6-3)}90$$

$$y=\frac{(x-1)(x-3)(x-6)}{(0-1)(0-3)(0-6)}18+\frac{(x-0)(x-3)(x-6)}{(1-0)(1-3)(1-6)}10$$

$$+\frac{(x)(x-1)(x-6)}{(3-0)(3-1)(3-6)}(-18)+\frac{(x)(x-1)(x-3)}{(6)(6-1)(6-3)}90$$

$=2x^3-10x^2+18$

$\therefore \frac{dy}{dx}=6x^2-20x$

$\therefore Slope\ of\ curve\ at\ x=2\ is\ 6(2)^2-20(2)=-16$

UNIT - II

NUMERICAL TECHNIQUES

CURVE FITTING

Method of Least Squares:

Suppose that a data is given in two variables x & y the problem of finding an analytical expression of the form $y = f(x)$ which fits the given data is called curve fitting.

Let $(x_1, y_1), (x_2, y_2)........(x_n, y_n)$ be the observed set of values in an experiment and $y = f(x)$ be the given relation $x \& y, Let\ E_1, E_2, ...E_x$ are the error of approximations then we have

$$E_1 = y_1 - f(x_1)$$
$$E_2 = y_2 - f(x_2)$$
$$E_3 = y_3 - f(x_3)$$

$E_n = y_n - f(x_n)$ Where $f(x_1), f(x_2)..........f(x_n)$ are called the expected values of y corresponding to $x = x_1, x = x_2........x = x_n$

$y_1, y_2......y_n$ are called the observed values of y corresponding to $x = x_1, x = x_2........x = x_n$ the differences $E_1, E_2.....E_n$ between expected values of y and observed values of y are called the errors, of all curves approximating a given set of points, the curve for which $E = E_1^2 + E_2^2 + ...E_n^2$ is a minimum is called the best fitting curve (or) the least square curve, This is called the method of least squares (or) principles of least squares

I. FITTING OF A STRAIGHT LINE:-

Let the straight line be $y = a + bx \rightarrow (1)$

Let the straight line (1) passes through the data points

$(x_1, y_1), (x_2, y_2)......(x_n, y_n) i.e., (x_i, y_i), i = 1, 2....n$

So we have $y_i = a + bx_i \quad \rightarrow (2)$

The error between the observed values and expected values of $y = y_i$ is defined as

$$E_i = y_i - (a + bx_i), i = 1,2 \ldots \ldots . . n \ \rightarrow (3)$$

The sum of squares of these errors is

$E = \sum_{i=1}^{n} E_1^2 = \sum_{i=1}^{n} [y_i - (a + bx_i)]^2$ Now for E to be minimum $\frac{\partial E}{\partial a} = 0; \frac{\partial E}{\partial b} = 0$

These equations will give normal equations

$$\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2$$

The normal equations can also be written as

$$\sum y = na + b\sum x$$
$$\sum xy = a\sum x + b\sum x^2$$

Solving these equation for a, b substituting in (1) we get required line of best fit to the given data.

II. NON LINEAR CURVE FITTING

1. PARABOLA:-

Let the equation of the parabola is $y = a + bx + cx^2$ ——(1)

The parabola (1) passes through the data points

$(x_1, y_1), (x_2, y_2) \ldots\ldots\ldots (x_n, y_n), i.e., (x_1, y_1); i = 1,2 \ldots\ldots n$

We have $y_i = a + bx_i + cx_i^2 \longrightarrow (2)$

The error E_i between the observed an expected value of $y = y_i$ is defined as

$E_i = y_i - (a + bx_i + cx_i^2), i = 1,2,3 \ldots\ldots n \rightarrow (3)$

The sum of the squares of these errors is

$$E = \sum_{i=1}^{n} E_1^2 = \sum_{i=1}^{n} (y_i - a - bx_i - cx_i^2)^2 \rightarrow (4)$$

for E to be minimum, we have

$$\frac{\partial E}{\partial a} = 0, \frac{\partial E}{\partial b} = 0, \frac{\partial E}{\partial c} = 0$$

The normal equations can also be written as

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$
$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

Solving these equations for a, b, c and satisfying (1) we get required parabola of best fit

2. POWER CURVE:-

The power curve is given by $y = ax^b \rightarrow (1)$

Taking logarithms on both sides $log_{10}y = log_{10}a + blog_{10}x$

(or) Y = A+ bX → (2) $where\ Y = log_{10}y, A = log_{10}a\ and\ X = log_{10}x$

Equation (2) is a linear equation in X & Y

∴ The normal equations are given by

$$\Sigma Y = nA + b\Sigma X$$

$$\Sigma XY = A\Sigma X + b\Sigma x^2$$

From these equations, the values A and b can be calculated then a = antilog (A) substitute a & b in (1) to get the required curve of best fit.

3. EXPONENTIAL CURVE:- $(1)\; y = ae^{bx}$ $(2)\; y = ab^{x}$

1. $y = ae^{bx} \rightarrow (1)$

Taking logarithms on both sides $log_{10}y = log_{10} + bxlog_{10}e$

$(or) Y = A + BX \quad \rightarrow (2)$ Where $Y = log_{10}y, A = log_{10}a, B = blog_{10}e$ & $X = x$

Equation (2) is a linear equation in X and Y

So the normal equation are given by

$$\Sigma Y = nA + B\Sigma X$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2$$

Solving the equation for A & B, we can find

$$a = anti\log A \;\&\; b = \frac{B}{\log_{10} e}$$

Substituting the values of a and b so obtained in (1) we get

The curve of best fit to the given data.

2. $y = ab^{x} \rightarrow (1)$

Taking log on both sides $log_{10}y = log_{10}a + xlog_{10}b$

$$(or)\; Y = A + BX \quad \rightarrow (2)$$

Where $Y = log_{10}y, A = log_{10}a, B = blog_{10}b$ & $x = X$

The normal equation (2) are given by

$$\Sigma Y = nA + B\Sigma X$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2$$

Solving these equations for A and B we can find $a = anti\log A, b = anti\log B$

Substituting a and b in (1)

Problems:

1. By the method of least squares, find the straight line that best fits the following data

x	**1**	**2**	**3**	**4**	**5**
y	**14**	**27**	**40**	**55**	**68**

Solution: The values of $\Sigma x, \Sigma y, \Sigma x^2$ *and* Σxy are calculated as follows

x_i	y_i	x_i^2	$x_i y_i$
1	14	1	14
2	27	4	54
3	40	9	120
4	55	16	220
5	68	25	340

$\Sigma x_i = 15; \Sigma y_i = 204; \Sigma x_i^2 = 55 \; and \; \Sigma x_i \Sigma y_i = 748$

The normal equations are

$\Sigma y = na + b\Sigma x \longrightarrow (1) \quad \Sigma xy = a\Sigma x + b\Sigma x^2 \longrightarrow (2)$

Solving we get $a = 0, b = 13.6$

Substituting these values a & b we get

$y = 0 + 13.6x \Rightarrow y = 13.6x$

2. Fit a straight line y=a+bx from data

x	**0**	**1**	**2**	**3**	**4**
y	**1**	**1.8**	**3.3**	**4.5**	**6.3**

Solution: Let the required straight line be y=a+bx…(1)

x	y	x^2	xy
0	1	0	0
1	1.8	1	1.8
2	3.3	4	6.6
3	4.5	9	13.5
4	6.3	16	25.2
$\sum x = 10$	$\sum y$=16.9	$\sum x^2$=30	$\sum xy = 47.1$

Normal equations are

$$\sum y = na + b\sum x$$
$$\sum xy = a\sum x + b\sum x^2$$

Substitute in above we get

5a+10b=16.9

10a+30b=47.1

Solving we get a=0.72; b=1.33.

$\therefore$ *The straight line is* $y = 0.72 + 1.33x$

3. Fit a straight line y=a+bx from data

x	**0**	**5**	**10**	**15**	**20**
y	**7**	**-11**	**16**	**20**	**26**

Solution: Let the required straight line be y=a+bx…(1)

x	y	x^2	xy
0	7	0	0
5	-11	25	-55
10	16	100	160
15	20	225	300
20	26	400	520
$\sum x = 50$	$\sum y$=58	$\sum x^2$=750	$\sum xy$ =925

Normal equations are

$$\sum y = na + b\sum x$$
$$\sum xy = a\sum x + b\sum x^2$$

Substitute in above we get

5a+50b=58

50a+750b=925

Solving we get a=-2; b=1.36.

$\therefore$ *The straight line is* $y = -2 + 1.36x$

4. Fit a straight line y=a+bx from data

x	**0**	**5**	**10**	**15**	**20**	**25**
y	**12**	**15**	**17**	**22**	**24**	**30**

Solution: Let the required straight line be y=a+bx…(1)

x	y	x^2	xy
0 5 10 15 20 25	12 15 17 22 24 30	0 25 100 225 400 625	0 75 170 330 480 750
$\sum x = 75$	$\sum y$=120	$\sum x^2$=1375	$\sum xy$ =1805

Normal equations are

$$\sum y = na + b\sum x$$
$$\sum xy = a\sum x + b\sum x^2$$

Substitute in above we get

6a+75b=58

75a+1375b=1805

Solving we get a=11.2862; b=0.6971.

$\therefore$ *The straight line is* $y = 11.2862 + 0.6971x$

5. Fit a straight line and a parabola to the following data and find out which one is most appropriate. Give your reason for the conclusion

x	**1**	**2**	**3**	**4**	**5**
y	**4**	**3**	**6**	**7**	**11**

Solution: Let the required straight line be y=a+bx…(1)

x	y	x^2	x^3	x^4	xy	x^2y
1 2 3 4 5	4 3 6 7 11	1 4 9 16 25	1 8 27 64 125	1 16 81 256 625	4 6 18 28 55	4 12 54 112 275
15	$\sum y$=31	$\sum x^2$=55	$\sum x^3$=225	$\sum x^4$=979	$\sum xy$ =111	$\sum x^2$y=457

Normal equations for fitting a straight line are

$$\sum y = na + b\sum x$$
$$\sum xy = a\sum x + b\sum x^2$$

Substitute in above we get

5a+15b=31

15a+55b=111

Solving we get a=0.8 ; b=1.8.

$$\therefore \text{ the straight line is } y = 0.8 + 1.8x$$

Let the required parabola be $y=a+bx+cx^2$…(2)

Normal equations for fitting a parabola are

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$
$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

Substituting values, we get

5a+15b+55c =31

15a+55b+225c = 111

55a+225b+979c =457

Solving we get a=4.7998;b=-1.6284;c=0.5714

∴The parabola fit is $4.7998x^2-1.6284x+0.5714$

Conclusion: Clearly parabola fit is best fit because error is near to ZERO than linear fit.

y	Error of linear fit E=y-f(x)	Error parabola fit E=y-g(x)
4	1.4	0.2572
3	-1.4	-0.8286
6	-0.2	0.9428
7	-1	-0.4286
11	1.2	0.0572

6. Fit a second degree parabola to the following data

x	**0**	**1**	**2**	**3**	**4**
y	**1**	**5**	**10**	**22**	**38**

Solution: Equation of parabola $y = a + bx + cx^2 \rightarrow (1)$

Normal equations $\Sigma y = na + b\Sigma x + c\Sigma x^2$

$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$

$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \quad \longrightarrow (2)$

x	y	xy	x^2	x^2y	x^3	x^4
0	1	0	0	0	0	0
1	5	5	1	5	1	1
2	10	20	4	40	8	16
3	22	66	9	198	27	81
4	38	152	16	608	64	256

$\Sigma x = 10, \Sigma y = 76, \Sigma xy = 243, \Sigma x^2 = 30, \Sigma x^2 y = 851, \Sigma x^3 = 100, \Sigma x^4 = 354$

Normal equations

$$76 = 5a + 10b + 30c$$
$$243 = 10a + 30b + 100c$$
$$851 = 30a + 100b + 354c$$

Solving $a = 1.42, b = 0.26, c = 2.221$

Substitute in (1) $\Rightarrow y = 1.42 + 0.26x + 2.221x^2$

7. Fit a second degree parabola to the following data:

x: 0 1 2 3 4

$f(x)$: 1 1.8 1.3 2.5 6.3

Solution:

Let the equation of the parabola be Y= a + b x + c x^2 -----(1)

The normal equations are given by $\Sigma y = na + b\Sigma x + c\Sigma x^2$

$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$-----(2)

$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$

x	y	x^2	x^3	x^4	xy	x^2y
0	1.0	0	0	0	0	0
1	1.8	1	1	1	1.8	1.8
2	1.3	4	8	16	2.6	5.2
3	2.5	9	27	81	7.5	22.5
4	6.3	16	64	256	25.2	100.8
10	12.9	30	100	354	37.1	130.3

Since there are 5 pairs of values so n=5 substituting the above values in (2) we get

$$12.9 = 5a + 10b + 30c$$

$$37.1 = 10a + 30b + 100c$$

$$130.3 = 30a + 100b + 354c$$

Solving the above equations we get a = 14.2, b = -1.07, c = 0.55

Substituting the above values in (1) $y = 14.2 - 1.07x + 0.55x^2$

Which is the required equation of the parabola.

8. Fit a parabola $y = a + bx + cx^2$ to the data given below

x:	1	2	3	4	5
y:	10	12	8	10	14

Solution: Let the equation of the parabola be $Y = a + bx + cx^2$ -----(1)

The normal equations are given by $\Sigma y = na + b\Sigma x + c\Sigma x^2$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \text{-----(2)}$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

x	y	x^2	x^3	x^4	xy	x^2y
1	10	1	1	1	10	10
2	12	4	8	16	24	48
3	8	9	27	81	24	72
4	10	16	64	256	40	160
5	14	25	125	625	70	350
15	54	55	225	979	168	640

Since there are 5 pairs of values so n=5 substituting the above values in (2) we get

$$54 = 5a + 15b + 55c$$

$$168 = 15a + 55b + 225c$$

$$640 = 55a + 225b + 979c$$

Solving the above equations we get a =14, b =-3.6857 , c = 0.7142

substituting the above values in (1) $y = 14 - 3.6857x + 0.7142x^2$

which is the required equation of the parabola.

9. Fit a parabola of the form $y = ax^2 + bx + c$

x:	1	2	3	4	5	6	7
y:	2.3	5.2	9.7	16.5	29.4	35.5	54.4

Solution: Let the equation of the parabola be $y = ax^2 + bx + c$ -----(1)

The normal equations are given by

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \text{-----}(2)$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

Table for calculations:

x	y	x^2	x^3	x^4	xy	x^2y
1	2.3	1	1	1	2.3	2.3
2	5.2	4	8	16	10.4	20.8
3	9.7	9	27	81	29.1	87.3
4	16.5	16	64	256	66	264
5	29.4	25	125	625	147	735
6	35.5	36	216	1296	213	1278
7	54.4	49	343	2401	380.8	2665.6
28	153	140	784	4676	848.6	5053

Since there are 5 pairs of values so n=5 substituting the above values in (2) we get

153 = 7a +28b +140c

848.6 = 28a+140b+784c

5053 = 140a+784b+4676c

Solving the above equations we get a =2.3705, b =-1.0924, c = 1.1928

substituting the above values in (1) y = 1.1928 x^2-1.0924 x+2.3705

which is the required equation of the parabola.

10. Fit a curve $y = ax^b$ to the following data

x	1	2	3	4	5	6
y	2.98	4.26	5.21	6.10	6.80	7.50

Sol:- Let the equation of the curve be $y = ax^b \to (1)$

Taking log on both sides $\log y = \log a + b \log x$

(or) $Y = A + bX \quad \longrightarrow (2)$ Where $Y = \log y, A = \log a, X = \log x$

The Normal Equations are $\Sigma Y = nA + b\Sigma X$

$$\Sigma XY = A\Sigma X + b\Sigma X^2 \quad \longrightarrow (3)$$

x	$X = \log x$	y	$Y = log\ y$	XY	X^2
1	0	2.98	0.4742	0	0
2	0.3010	4.26	0.6294	0.1894	0.0906
3	0.4771	5.21	0.7168	0.3420	0.2276
4	0.6021	6.10	0.7853	0.4728	0.3625
5	0.6990	6.80	0.8325	0.5819	0.4886

$$\Sigma X = 2.8574, \Sigma Y = 4.3133, \Sigma XY = 2.2671, \Sigma X^2 = 1.7749$$

4.3313=6A+208574b and 2.2671=2.8574A+1.7749b

Solving A=0.4739, b=0.5143

A=anti log (A) =2.978

$$\therefore y = 2.978x^{0.5143}$$

11 . Fit a curve $y = ab^x$

x	**2**	**3**	**4**	**5**	**6**
y	**144**	**172.8**	**207.4**	**248.8**	**298.5**

Solution: Let the curve to be fitted is $y = ab^x$

Taking log on both sides $\log y = \log a + x \log b \quad \longrightarrow (1)$

$$Y = A + xB \quad \longrightarrow (2)$$

$$Y = \log y, A = \log a, B = \log b$$

$$\Sigma Y = nA + B\Sigma x$$

$$\Sigma xY = A\Sigma x + b\Sigma x^2 \quad \longrightarrow (3)$$

x	y	x^2	$Y = \log y$	xy
2	144.0	4	2.1584	4.3168
3	172.8	9	2.2375	6.7125
4	207.4	16	2.3168	9.2672
5	248.8	25	2.3959	11.9795
6	298.5	36	2.4749	14.8494

Substituting these values the normal equations are

11.5835 = 5A + 20B

47.1254 = 20A+90B

Soving A and B, taking antilogarithms

a=100, b=1.2

Substituting in (1), the equation of the curve is y =100(1.2)^x

NUMERICAL INTEGRATION

INTRODUCTION:

The process of evaluating a definite integral from a set of tabulated values of the integrand f(x), which is not known explicitly is called Numerical Integration.

Newton –Cote's Quadrature Formula:

We want to find Definite integral form $\int_a^b f(x)dx$, where f(x)is unknown explicitly,then We replace f(x) with interpolating polynomial.

Here we replace with Newton Forward Interpolation formula

Divide the interval (a,b) into n sub intervals of width h so that

a $=x_0 < x_1 = x_0 + h \ldots\ldots\ldots < x_n = x_n + h = b$ Then

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \ldots\ldots\ldots + \frac{p(p-1)(p-2)\ldots\ldots(p-(n-1))}{n!}\Delta^n y_0$$

Where p=$\frac{x-x_0}{h}$ hdp=dx a t x=$x_0 \Rightarrow$ p=0 and x=$x_n \Rightarrow$ p=n

$$\therefore \int_a^b f(x)dx = \int_{xo}^{xn} y_n(x)\,dx = h\int_{xo}^{xn}\left(y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \ldots\ldots\ldots\right)dp$$

$$= h\int_0^n\left(y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \ldots\ldots\ldots\right)dp$$

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n}{12}(2n-3)\Delta^2 y_0 + \frac{n}{24}(n-2)^2\Delta^3 y_0 + \ldots\right]$$

This is Newton Cotes Quadrature Formula.

Derive Trapezoidal Rule for numerical integration of $\int_a^b f(x)dx$:

I.TRAPEZOIDAL RULE:-

Sub n=1 in Newton Cotes Quadrature_Formula and taking the curve y = f(x) passing through (x_0, y_0) *and* (x_1, y_1) as a straight line so that differences of order higher than first become zero(i.e., Δ^2, Δ^3 *etc become zero*) (n=number of intervals)

$$\int_{x0}^{x1} f(x)dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right] = h/2[y_0+y_1]\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

Similarly we get

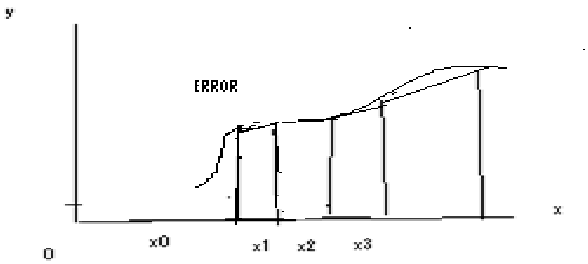
$$\int_{x1}^{x2} f(x)dx = h/2[y_1+y_2]\ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

Adding above we get

$$\int_{x_0}^{x_n} ydx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + ----- + y_{n-1})]$$

$$\int_{x_0}^{x_n} ydx = \frac{h}{2}[(sum\ of\ the\ Ist\ \&\ last\ oridinates) + 2(sum\ of\ the\ remaining\ ord.)]$$

Geometrical interpretation of Trapezoidal Rule:



Here trapezoidal rule denotes sum of areas of above trapeziums.

Derive Simpson's 1/3 Rule for numerical integration of $\int_a^b f(x)dx$

II. Simpson's 1/3 Rule (n=2)

sub n=2 in Newton Cotes Quadrature Formula and taking the curve y = f(x) passing through $(x_0, y_0), (x_1, y_1)$ *and* (x_2, y_2) as a parabola so that differences of order higher than second become zero(i.e., Δ^3, Δ^4 *etc become zero*)

$$\int_{x_0}^{x_2} f(x)dx = 2h[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0]$$

We know E = 1+Δ

Δ

then $\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}[y_0 + 4y_1 + y_2]$

Similarly $\int_{x_2}^{x_4} f(x)\, dx = 2h[y_2 + 4y_3 + y_4]$

----

and so on $\int_{x_{n-2}}^{x_n} f(x)dx = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$

Adding

$$\int_{x_0}^{x_n} ydx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_2 + .. + y_{n-1}) + 2(y_2 + y_4 + .. + y_{n-2})]$$

$$\int_{x_0}^{x_n} ydx = \frac{h}{3}[(sum\ of\ the\ first\ and\ last\ oridinates) + 4(sum\ of\ the\ odd\ ordinates)$$
$$+ 2(sum\ of\ the\ remaining\ even\ ordinates)]$$

This is known as Simpson's $^1/_3$ Rule (or) Simply Simpson's Rule. .

III. Simpson's 3 / 8 Rule

$\int_{x_0}^{x_n} y dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + .. + y_{n-1}) + 2(y_3 + y_6 + .. + y_{n-3})]$

Note: -

1. Trapezoidal Rule is applicable for any number of subintervals
2. Simpson's $^1/_3$ rule is applicable when the number of subintervals must be even
3. Simpson's $^3/_8$ rule is applicable when the number of subintervals must be multiple of 3

Compare Trapezoidal Rule and Simpson's 1/3 rule

In trapezoidal rule we take n=1(no of subintervals).between every two points we are taking a st line(LINEAR) where as in simpsons rule n=2 means We are taking a PARABOLASo error is less compare to trapezoidalrule.

PROBLEMS

1. Evaluate $\int_0^{\pi} \frac{\sin x}{x}$dx by using trapezoidal and simpson's 1/3 rules taking n=6

SOL: h= $\frac{b-a}{n} = \frac{\pi}{6}$

Here $\frac{\sin 0}{0}$=1 since $\lim_{x \to 0} \frac{\sin x}{x}$=1

x	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	π
sinx	0	0.5	0.866	1	0.866	0.5	0
Sinx/x	1	0.9549	0.8270	0.6366	0.4135	0.1910	0

i) Trapezoidal rule :

$\int_0^1 \frac{1}{1+x}$dx =$\frac{h}{2}$[(sum of first and last ordinates) + 2(sum of the remaining ordinates)]

=$\frac{\pi}{12}$ [(1+0) +2(0.827+0.4135+0.9549+0.6366+0.1910)] = 1.8446

ii) Simpson's 1/3 rule:

$\int_0^1 \frac{1}{1+x}$dx= $\frac{h}{3}[(sum\ of\ the\ Ist\ \&\ last\ oridinates) + 4(sum\ of\ the\ odd\ ordinates) +$

$2(sum\ of\ the\ remaining\ even\ ordinates)]$

= $\frac{\pi}{18}$[(1+0)+2(0.827+0.4135)+4(0.9549+0.6366+0.1910)] = 1.852

x	0	1/6	2/6	3/6	4/6	5/6	6/6
y=$\frac{1}{1+x}$	1 y_0	0.8571 y_1	0.75 y_2	0.6666 y_3	0.6 y_4	0.5454 y_5	0.5 y_6

2. Evaluate $\int_0^1 \frac{1}{1+x}dx$ by using trapezoidal , simpson's 1/3, Simpson's 3/8 rules.

SOL: We want to use above 3 rules so take n=6

$$h= \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$$

i) Trapezoidal rule :

$\int_0^1 \frac{1}{1+x}dx = \frac{h}{2}$[(sum of first and last ordinates) + 2(sum of the remaining ordinates)]

$=\frac{1}{2}$ [(1+0.5) +2(0.8571+0.5454+0.75+0.6+0.6666)] = 0.69485

ii) Simpson's 1/3 rule:

$$\int_0^1 \frac{1}{1+x}dx=\frac{h}{3}[(sum\ of\ the\ first\ and\ last\ oridinates) + 4(sum\ of\ the\ odd\ ordinates) + 2(sum\ of\ the\ remaining\ even\ ordinates)]$$

$=\frac{1}{18}$[(1+0.5)+2(0.75+0.6)+4(0.8571+0.6666+0.5454)] = 0.6931

iii) Simpson's 1/3 rule:

$$\int_0^1 \frac{1}{1+x}dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 +..+ y_{n-1}) + 2(y_3 + y_6 + ..+ y_{n-3})]$$

$= \frac{1}{16}$[(1+0.5)+2(0.6666)+3(0.8571+0.75+0.6+0.5454)] = 0.6932

3. Evaluate $\int_4^{5.2} logx$ dx by using trapezoidal , simpson's 1/3,Simpsons 3/8 rules from

x	4	4.2	4.4	4.6	4.8	5	5.2
logx	1.3863 y_0	1.4351 y_1	1.4816 y_2	1.5261 y_3	1.5686 y_4	1.6094 y_5	1.6487 y_6

SOL: Here h=4.2-4=0.2

i) **Trapezoidal rule :**

$\int_4^{5.2} logx$ dx $=\frac{h}{2}$[(sum of first and last ordinates) + 2(sum of the remaining ordinates)]

$=\frac{0.2}{2}$[(1.3863+1.6487)+2(1.4351+1.4816+1.5261+1.5686+1.6094)] = 1.8277

ii) Simpson's 1/3 rule:

$$\int_0^1 \frac{1}{1+x}dx=\frac{h}{3}[(sum\ of\ the\ first\ and\ last\ oridinates) + 4(sum\ of\ the\ odd\ ordinates) + 2(sum\ of\ the\ remaining\ even\ ordinates)]$$

$=\frac{0.2}{3}$[(1.3863+1.6487)+2(1.4816+1.5686)+4(1.4351+1.5261+1.6094)] =1.8279

iii) Simpson's 3/8 rule:

$$\int_0^1 \frac{1}{1+x}dx= \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 +..+ y_{n-1}) + 2(y_3 + y_6 +..+ y_{n-3})]$$

$=\frac{0.6}{18}$[(1.3863+1.6487)+2(1.5261)+3(1.4351+1.4816+1.5686+1.6094)] = 1.8278

4. The velocity v (m/sec) of a particle at distance S(m) from a point on its path given by following table

S	0	10	20	30	40	50	60
v	47	58	64	65	61	52	38

Estimate the time taken to travel 60 meters by Simpsons 1/3 and 3/8 rules.

SOL: Let $v = \frac{dv}{dt}$ be the velocity of particle at any time 't'

Then $dt = \frac{ds}{v}$ Integrating on both sides with limits 0 to 60

Then $t = \int_0^{60} \frac{1}{v}$ ds

S	0	10	20	30	40	50	60
v	47	58	64	65	61	52	38
1/v	0.0212	0.0172	0.0156	0.0153	0.0163	0.0192	0.0263

I) Simpson's $\frac{1}{3}$rule:

$$\int_0^{60} \frac{1}{v}\, ds = \frac{h}{3}\begin{bmatrix} (sum\ of\ the\ first\ and\ last\ oridinates) \\ +4(sum\ of\ the\ odd\ ordinates) \\ +2(sum\ of\ the\ remaining\ even\ ordinates) \end{bmatrix}$$

$=\frac{10}{3}$[(0.0212+0.0263)+2(0.0156+0.0163)+4(0.0172+0.0153+0.0192)]= **1.0603 sec**

ii) Simpson's $\frac{3}{8}$ rule:

$$t = \int_0^{60} \frac{1}{v}\, ds = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + . + y_{n-1}) + +2(y_3 + y_6 + . + y_{n-3})\right]$$

$=\frac{30}{8}$[(0.0212+0.0263)+2(0.0153)+3(0.0172+0.0163+0.0192)]= **0.8857 sec**

5.Evaluate $\int_0^{\pi/2} e^{sinx}$dx correct to four decimals places by Simpsons 3/8rule

SOL: Here h=$\frac{\pi}{12}$

X	0	$\frac{\pi}{12}$	$\frac{2\pi}{12}$	$\frac{3\pi}{12}$	$\frac{4\pi}{12}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
Y	1	1.2954	1.6487	2.0281	2.3774	2.6272	2.718

Simpson's $\frac{3}{8}$ rule:

$$t = \int_0^{60} \frac{1}{v}\, ds = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + . + y_{n-1}) + +2(y_3 + y_6 + . + y_{n-3})\right]$$

$=\frac{3\pi}{96}$[(1+2.718)+2(2.0281)+3(1.2954+1.6487+2.3774+2.6272)=**3.1015**

6. Evaluate $\int_0^1 \frac{1}{1+x^2}dx$ using Simpson's 3/8 rule

Sol: Divide the interval into 6 sub intervals & tabulate the values **of $f(x_i) = \frac{1}{1+x^2}$** as follows

x_i	0	1/6	2/6	3/6	4/6	5/6	6/6
$f(x_i)$	1	0.9729	0.90	0.80	0.69231	0.59016	0.5

Here $h = 1/6$

Using Simpson's rule

$$I = \int_0^1 \frac{1}{1+x^2}dx = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8.6}[(1.0 + 0.50) + 3(0.9729 + 0.90 + 0.69231 + 0.59016) + 2(0.80)] = \frac{1}{16}(12.5662)$$

$$= 0.785395 \cong 0.7854$$

7. **Find the value of $\int_0^1 \frac{1}{1+x^2}dx$, taking 5 sub internals & by using Trapezoidal rule.**

Sol:

$$f(x) = \frac{1}{1+x^2}, n = 5, a = 0, b = 1$$

$$\therefore h = \frac{b-a}{n} = \frac{1-0}{5} = 0.2$$

Construct a table of values of $x_i \& y_i = f(x_i)$ as follows

x_i	0.0	0.2	0.4	0.6	0.8	1.0
y_i	1.00	0.961538	0.832069	0.735294	0.609755	0.50

Using Trapezoidal rule we get

$$I = \int_0^1 \frac{1}{1+x^2}dx = \frac{0.2}{2}\left[(1.0 + 0.50) + 2(0.961538 + 0.832069 + 0.735294 + 0.609759)\right]$$

$= 0.783734$

8. Find the area bounded by the curve $f(x) = y$ and x-axis from $x = 7.47$ to $x = 7.52$

x_i	7.47	7.48	7.49	7.50	7.51	7.52
y_i	1.93	1.95	1.98	2.01	2.03	2.06

Sol:- Here h = 0.01

Area formed by the curve $y = f(x)$ and x – axis from $x = 7.47$ to $x = 7.52$ is

$$Area = \int_{7.47}^{7.52} f(x)dx$$

Applying Trapezoidal rule we get

$$Area = \int_{7.47}^{7.52} f(x)dx = \frac{h}{2}\left[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)\right]$$

$$= \frac{0.01}{2}\left[(1.93 + 2.06) + 2(1.95 + 1.98 + 2.01 + 2.03)\right]$$

$$= 0.0996$$

3. **Evaluate 9.Evaluate 9.Find** $\int_0^1 x^3 dx$ **with 5 sub intervals by Trapezoidal rule**

Sol: - Here $a = 0, b = 1, n = 5 \; \& \; y = f(x) = x^3$

$$\therefore h = \frac{b-a}{n} = \frac{1-0}{5} = 0.2$$

The values of x & y are tabulated below

x	0.2	0.4	0.6	0.8	1
y	0.008	0.064	0.216	0.512	1

By Trapezoidal rule

$$\int_0^1 x^3 dx = \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.2}{2}[(0.008 + 1) + 2(0.064 + 0.216 + 0.512)]$$

$$= 0.2592 \cong 0.26$$

10. Evaluate $\int_0^{\pi} t \sin t dt$ **using Trapezoidal rule**

Sol:- Divide the interval $(0, \pi)$ in to 6 parts each of width $h = \frac{\pi}{6}$

The values of $f(t) = t \sin t$ are given below

t	0	$\pi/6$	$2\pi/6$	$3\pi/6$	$4\pi/6$	$5\pi/6$	π
$f(t) = y$	0 y_0	0.2618 y_1	0.9069 y_2	1.5708 y_3	1.8138 y_4	1.309 y_5	0 y_6

By Trapezoidal rule

$$\int_0^{\pi} t \sin t dt = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{\pi}{12}\left[(0+0) + 2(0.2618 + 0.9069 + 1.5708 + 1.8138 + 1.309)\right]$$

$$= \frac{\pi}{12}(11.7246)$$

$$= 3.0695 \cong 3.07$$

11. Find the value of $\int_1^2 \frac{dx}{x}$ by Simpson's 1/3 rule. Hence obtain approx. value of $log_e 2$

Sol:- Divide the interval (1,2) in to 8(**even**) parts each of width $h = 0.125$

x	1	1.125	1.25	1.375	1.5	1.625	1.75	1.875	2
$y = \frac{1}{x}$	1 y_0	0.8888 y_1	0.8 y_2	0.7272 y_3	0.6666 y_4	0.6153 y_5	0.5714 y_6	0.5333 y_7	0.5 y_8

By Simpson's 1/3 rule

$$\int_1^2 \frac{dx}{x} = \frac{h}{3}[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$= \frac{0.125}{3}\left[(1+0.5)+4(0.8888+0.7272+0.6153+0.5333)+2(0.8+0.6666+0.5714)\right]$$

$$= \frac{0.125}{3}[1.5+11.0584+4.076] \qquad = \frac{0.125}{3}[16.6344] \qquad = 0.6931$$

By actual integration,

$$\int_1^2 \frac{dx}{x} = [\log x]_1^2 = \log 2 - \log 1 = \log 2$$

Hence $\log 2 = 0.6931$, correct to four decimal places

12. A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given in the table below. Using Simpson's 1/3 rule, find the velocity of the rocket at t = 80 seconds

t (sec)	0	10	20	30	40	50	60	70	80
$f\left(cm/\sec^2\right)$	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

Sol:- We know that the rate of velocity is acceleration I.e., $f = \frac{\partial v}{\partial t}$

$\therefore$ Velocity of the rocket at $t = 80\sec$ is given

$$v = \int_0^{80} f dt$$

$$= \frac{10}{3}[(30 + 50.67) + 4(31.63 + 35.47 + 40.33 + 46.69)2(33.34 + 37.75 + 43.25)]$$

$$= \frac{10}{3}[80.67 + 616.48 + 228.68] = \frac{10}{3}(925.83) = 3086.1$$

13. A river is soft wide. The depth 'd' in feet at a distance x ft from one bank is given by the table

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	15	14	8	3

Find approximately the area of cross-section

Sol:- Here $h = 10, y_0 = 0, y_1 = 4, y_2 = 7, y_3 = 9, y_4 = 12, y_5 = 15, y_6 = 14, y_7 = 8 \& y_8 = 3$

Area of cross section = $\int_0^{80} y\,dx$

Area = $\frac{h}{3}\left[(y_0+y_8)+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)\right]$

$$=\frac{10}{3}\left[(0+3)+4(4+9+15+8)+2(7+12+14)\right]$$

$$=\frac{10}{3}[3+144+66]$$

$$=710\,sq.ft$$

14. Evaluate $\int_0^{\pi}\sin x\,dx$ by dividing the interval $(0,\pi)$ in to 8 sub intervals & using Simpson's 1/3 rule

Sol: - Given $a=0, b=\pi, n=8\ \&\ f(x)=\sin x$

$$\therefore h=\frac{b-a}{n}=\frac{\pi-0}{8}=\pi/8$$

Tabulate the values of $\sin x$ as follows

x_i	0	$\pi/8$	$\pi/4$	$3\pi/8$	$\pi/2$	$5\pi/8$	$6\pi/8$	$7\pi/8$	π
$\sin x_i$	0	0.38	0.71	0.92	1	0.92	0.710	0.38	0

Simpson's 1/3 rule for n = 8 is

$$I=\int_a^b f(x)\,dx=\frac{h}{3}\left[(y_0+y_8)+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)\right]$$

$$=\frac{\pi}{8.3}\left[(0+0)+4(0.38+0.92+0.92+0.38)+2(0.71+1.0+0.71)\right]$$

$$=1.99$$

15. Find the area bounded by the curve $y=e^{-x^2/2}$, x axis between $x=0\ \&\ x=3$ by using Simpson's 3/8 rule

Sol:- Divide the interval $(0,3)$ in to 6 sub intervals $\therefore h=\frac{3-0}{6}=0.5$

The values of $y_i=e^{-x^2/2}$ are tabulated as follows

x_i	0.0	0.5	1.0	1.5	2.0	2.5	3.0
$y(x_i)$	1.0	1.33	1.649	3.080	7.389	22.760	90.017

By Simpson's 3/8 rule we get

$$I=\int_0^1 e^{-x^2/2}\,dx=\frac{3h}{8}[(y_0+y_6)+3(y_1+\ y_2+y_4+y_5)+2y_3]$$

$$=\frac{3(0.5)}{8}[(1.00+90.017)+3(1.33+1.649+7.389+22.760)+2(3.080)]$$

$$=36.744\ \text{Square units}$$

Numerical solutions of ordinary differential equations

The important methods of solving ordinary differential equations of first order numerically are as follows

1) Picard's method
2) Taylor's series method
3) Euler's method
4) Modified Euler's method of successive approximations
5) Taylor's series method
6) Runge- Kutta method

To describe various numerical methods for the solution of ordinary differential equations, we consider the general 1st order differential equation.

$\frac{dy}{dx}$=f(x,y)-------(1) with the initial condition $y(x_0)=y_0$

The methods will yield the solution in one of the two forms:

i) A series for y in terms of powers of x, from which the values of y can be obtained by direct substitution.

ii) A set of tabulated values of y corresponding to different values of x

The methods of Taylor and Picard belong to class (i)

The methods of Euler, Runge - Kutta method, Adams, Milne etc, belong to class (ii)

Picard's method of successive approximations

Consider the following differential equation $\frac{dy}{dx}$=f(x,y)-------(1) initial condition is that

$y=y_0$ at $x=x_0$---- (2)

the equation is dy=f(x,y)dx

Integrating the equation between the limits x_0 and x we get

$$\int_{x=x_0}^{x} dy = \int_{x_0}^{x} f(x, y)dx$$

i.e., $[y]_{x=x_0}^{x} = \int_{x_0}^{x} f(x, y)dx$

i.e., $y(x)-y(x_0) = \int_{x_0}^{x} f(x, y)dx$

or $y(x)=y_0+\int_{x_0}^{x} f(x,y)dx$ ----(3)

We find that the R.H.S of (3) contains the unknown y under the integral sign. An equation of this kind is called an integral equation and it can be solved by a process of successive approximations.

Picard's method gives a sequence of functions $y^1(x)$, $y^{(2)}(x)$, $y^{(3)}(x)$,….

Which form a sequence of approximation to y converges to y(x)

To get the 1st approximation $y^{(1)}(x)$, put $y = y_0$, in the integral of (3)

We get $y^{(1)}(x) = y_0 + \int_{x0}^{x} f(x,y_0)dx$ ------------ (4)

Since $f(x,y_0)$ is a function of x, it is a possible to integral it with respect to x

To get the 2nd approximation $y^{(2)}(x)$ for y, put $y = y^{(1)}(x)$ in the integral of (3) we get

$$y^{(2)}(x) = y_0 + \int_{x0}^{x} f(x, y^{(1)}(x))dx \to (5)$$

Similarly, a 3rd approximation of $y^{(3)}$ for y is $y^{(3)} = y_0 + \int_{x_0}^{x} f[x, y^{(2)}(x)]dx \longrightarrow (6)$

Proceeding in this way, we get the n^{th} approximation $y^{(n)}(x)$ for y as

$$y_n(x) = y_0 + \int_{x_0}^{x} f\left[x, y^{(n-1)}(x)\right]dx \to (7) \text{ Or } y_n = y_0 + \int_{x_0}^{x} f(x, y^{n-1})dx,\ n = 1,2 \ldots\ldots.$$

Eqn 7 gives the general iterative formula for y iterations are repeated until the two successive approximations $y^{(i)}$ and $y^{(i-1)}$ are sufficiently close.

Eqn 7 is known as picards iteration formula.

This method is not convenient for computer based solutions

PROBLEMS:

1. Obtain y(0.1) given $y' = \frac{y-x}{y+x}, y(0)=1$ **by picad's method.**

SOL: Here f(x,y)=. $\frac{y-x}{y+x}$, $y_0=1$ and $x_0=0$.

By picard's method, a sequence of successive approximation to y are given by

$$y^{(n)}(x) = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})dx \text{ , n=1,2,3,---}$$

$$y^{(n)}(x) = 1 + \int_{0}^{x} f(x, y^{(n-1)})dx \text{ , n=1,2,3,---}$$

First approximation: we have

$$y^{(1)}(x) = 1+\int_0^x f(x,y^0)dx = 1+\int_0^x f(x,1)dx$$

$$= 1+\int_0^x \frac{1-x}{1+x}\,dx$$

$$= 1+\int_0^x (\frac{2}{1+x} - 1)\,dx$$

$$= 1+1 + [-x + 2\log(1+x)]_0^x$$

= 1+[-x+2log(1+x)]-(0+2log(0))

= 1-x+2log(1+x)

Second approximation, we have

$$y^{(2)} = 1 + \int_0^x f\left(x, y^{(1)}\right)dx$$

$$= 1+\int_0^x \frac{1-x+2\log(1+x)-x}{1-x+2\log(1+x)+x}\,dx$$

$$= 1+\int_0^x [\frac{-2x}{1+2\log(1+x)} + 1]dx$$

It is clear that the resulting expressions too big as we proceed to higher approximations.

Hence approximate value of y(x) is $y^{(1)}(x)$=y(0.1)= 1-x+2log(1+x)=1-0.1+2log(1.1) **= 1.0906**

2. Use picard's method to approximate y when x=0.2 G.T. y=1 when x=0 and $\frac{dy}{dx} = x - y$

Sol: Consider $\frac{dy}{dx} = f(x, y)$ where y = y_0 at x = x_0.

Here f(x,y) = x-y, x_0=0 and y_0=1.

By picard's method, picard's iteration formula is

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})dx$$

$$\therefore y^{(n)} = 1 + \int_0^x f(x, y^{(n-1)})dx \rightarrow (1)$$

First approximation: put y=1 on R.H.S at (1).

$$y^{(1)} = 1 + \int_0^x f(x,1)dx = 1 + \int_0^x (x-1)dx = 1 + \left[\frac{x^2}{2} - x\right]_0^x = 1 - x + \frac{x^2}{2}$$

Second approximation:

$$y^{(2)} = 1 + \int_0^x f\left(x, y^{(1)}\right)dx = 1 + \int_0^x f\left(x, 1 - x + \frac{x^2}{2}\right)dx$$

$$= 1 + \int_0^x \left[x - \left(1 - x + \frac{x^2}{2}\right)\right] dx$$

$$= 1 + \int_0^x \ (2x - 1 - \frac{x^2}{2})]dx = 1 + x^2 - x - \frac{x^3}{6}$$

Third approximation:

$$y^{(3)} \ = 1 + \int_0^x f\left(x, y^{(2)}\right)dx = 1 + \int_0^x x - \left(1 + x^2 - x - {x^3}/{6}\right)$$

$$= 1 + \int_0^x \ (x - 1 - x^2 + x + \frac{x^3}{6})dx = 1 + x^2 - x - \frac{x^3}{3} + \frac{x^4}{24}$$

Fourth approximation:

$$y^{(4)} = 1 + \int_0^x f\left(x, y^{(3)}\right)dx = 1 + \int_0^x x - \left(1 + x^2 - x - \frac{x^3}{3} + \frac{x^4}{24}\right)dx$$

$$= 1 + \int_0^x \ (x - 1 - x^2 + x + \frac{x^3}{3} - \frac{x^4}{24})dx$$

$$= 1 + \frac{x^2}{2} - x - \frac{x^3}{3} + \frac{x^2}{2} + \frac{x^4}{12} - \frac{x^5}{120}$$

$$= 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120}$$

Fifth approximation:

$$y^{(5)} = 1 + \int_0^x f(x, y^{(4)})dx = 1 + \int_0^x [x - y^{(4)}]dx$$

$$= 1 + \int_0^x \ (x - 1 + x - x^2 + \frac{x^3}{3} - \frac{x^4}{12} + \frac{x^5}{120})dx$$

$$= 1 + \int_0^x \ (2x - 1 - x^2 + \frac{x^3}{3} - \frac{x^4}{12} + \frac{x^5}{120})dx$$

$$= 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} - \frac{x^6}{720}$$

When x=0.2, we have

$y_0=1$, $y^{(1)}=0.82$, $y^{(2)}=0.83867$, $y^{(3)} = 0.83740$, $y^{(4)} = 0.83746$ and $y^{(5)} = 0.83746$

y=0.83746 at x=0.2

3. Find an approximate value of y for x=0.1, x=0.2, if $\frac{dy}{dx} = x + y$ and y=1 at x=0 using picard's method. Check your answer with the exact particular solution.

Sol: Consider $\frac{dy}{dx} = f(x, y)$ where y = y_0 at x = x_0.

Here f(x,y) = x+y, x_0=0 and y_0=1.

By picard's method, a sequence of successive approximations

are given by. $y^{(n)}(x) = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)}(x))dx$

(or)

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})dx$$

For n=1,2,3,---- (1)

$$y^{(n)} = 1 + \int_{0}^{x} f(x, y^{(n-1)})dx$$

For n=1,2,3,--- (2)

when x=0.1

<u>First approximation</u>

$$y^{(1)} = 1 + \int_0^1 (x+1)dx = 1 + x + \frac{x^2}{2}$$

<u>Second approximation</u>

$$y^{(2)} = 1 + \int_0^x x + \left(1 + x + \frac{x^2}{2}\right) dx$$

$$= 1 + \int_0^x \left(1 + 2x + \frac{x^2}{2}\right) dx = 1 + x + x^2 + {x^3}/{6}$$

<u>Third approximation</u>

$$y^{(3)} = 1 + \int_0^x \left[x + \left(1 + x + x^2 + {x^3}/{6}\right)\right] dx$$

$$= 1 + \int_0^x \left(1 + 2x + x^2 + {x^3}/{6}\right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

y=0.1

$$y^{(3)} = 1 + (0.1) + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{24}$$

$$= 1.1 + 0.01 + \frac{(0.001)}{3} + \frac{0.0001}{24}$$

= 1.1+0.01+0.0003+0.0000041

= 1.1103041 ~ 1.1103

y=0.2

$$y^{(3)} = 1 + (0.2) + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{24}$$

= 1.2+0.04+0.00266+0.0000666 = 1.2427

y=1.1103 at x=0.1 and y=1.2427 at x=0.2

Analytical solution:

The exact solution of $\frac{dy}{dx} = x + y$, y(0)=1 can be found as follows.

The equation can be written as $\frac{dy}{dx} - y = x$

This is a linear equation in y [i.e, $\frac{dy}{dx} + p.y = Q$]

then p=-1, Q. $I.F = e^{\int pdx} = e^{\int(-1)dx} = e^{-x}$

General solution is y . I.F= $\int QXI.Fdx + c$ (X means multiplication)

$y.e^{-x} = \int x.e^{-x}dx + c$

$y.e^{-x} = -e^{-x}(x+1)+c$. or $y = -(x+1)+ce^{+x}$

when x=0, y=1 i.e, i=-(0+1)+c or c=2

Hence the particular solution of the equation is

$y=-(x+1)+2e^x=2e^x-x-1$.

For x=0.1, y = $e^{0.1}$-0.1-1=2(1.1052)-0.1-1=1.1104

For x=0.2, y= $2e^{0.2}$-0.2-1 = 2(1.2214)-0.2-1=1.2428.

4. Find the value of y for x=0.4 by picard's method, given that

$\frac{dy}{dx} = x^2 + y^2$, **y(0)=0.**

Sol: Consider $\frac{dy}{dx} = f(x, y)$ and $y=y_0$ at $x=x_0$ i.e., $y(x_0)=y_0$

Here $f(x, y) = x^2 + y^2$ and $x_0=0$, $y_0=0$.

By picard's method, the successive approximation are given by

$$y^{(n)}(x) = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})dx \text{ , n=1,2,3,---}$$

$$y^{(n)}(x) = 0 + \int_{0}^{x} f(x, y^{(n-1)})dx$$

$$y^{(n)}(x) = \int_{0}^{x} f(x, y^{(n-1)})dx \text{ , n=1,2,3-----} \quad (1)$$

The first approximation:

$$y^{(1)}(x) = \int_{0}^{x} f(x, y^{(0)})dx = \int_{0}^{x} f(x,0)dx = \int_{0}^{x} x^2 dx = \frac{x^3}{3}$$

The second approximation:

$$y^{(2)}(x) = \int_{0}^{x} f(x, y^{(1)})dx = \int_{0}^{x} f[x^2 + (\frac{x^3}{3})^2]dx = \frac{x^3}{3} + \frac{x^7}{54}$$

Calculation of $y^{(3)}$ is tedions and hence approximate value is $y^{(2)}$.

For x=0.4, $y^{(1)} = \frac{(0.4)^3}{3}$ =0.02133

$y^{(2)} = \frac{(0.4)^3}{3} + \frac{(0.4)^7}{54}$ =0.0213333+0.0000303.

=0.0213636 ~ 0.0214(correct to 4 decimal places)

y=0.0214 at x=0.4.

5. Given that $\frac{dy}{dx} = 1 + xy$ and y(0)=1, compute y(0.1) and y(0.2) using picard's method.

Sol: Consider $\frac{dy}{dx} = f(x, y)$ and $y(x_0)=y_0$.

Here f(x,y)=1+xy and $y_0=1$, $x_0=0$.

By picard's method, the successive approximations are given by

$$y^{(n)}(x) = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})dx \text{ , n=1,2,3,---}$$

$$y^{(n)}(x) = 1 + \int_{0}^{x} f(x, y^{(n-1)})dx \text{ , n=1,2,3---}$$

The first approximation:

$$y^{(1)}(x) = 1 + \int_0^x f(x, y^{(0)})dx = 1 + \int_0^x f(x,1)dx = 1 + \int_0^x (1+x)dx = 1 + x + \frac{x^2}{2}$$

The second approximation:

$$y^{(2)}(x) = 1 + \int_0^x f\left(x, y^{(1)}\right)dx = 1 + \int_0^x \left[(1+x)\left(1 + x + \frac{x^2}{2}\right)\right]dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

The third approximation:

$$y^{(3)}(x) = 1 + \int_0^x f(x, y^{(2)})dx = 1 + \int_0^x [1 + x(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8})]dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}$$

It is clear that the resulting expressions too big, as we proceed to higher approximations. Hence approximative value is $y^{(3)}$.

For x=0.1,

$$y^{(3)} = 1 + (0.1) + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{8} + \frac{(0.1)^5}{15} + \frac{(0.1)^6}{48}$$

=1+0.1+0.005+0.000333+0.0000125+0.000000666+0.00000002

$=1.105346 \cong 1.10535$

y(0.1)=1.10534.

For x=0.2,

$$y^{(3)} = 1 + (0.2) + \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{8} + \frac{(0.2)^5}{15} + \frac{(0.2)^6}{48}$$

= 1.2+0.02+0.0026666+0.0002+0.00002133+0.000001333

= 1.222889 ~ 1.22289

y(0.2)=1.22289.

6. Using picard's method, obtain the solution of $\frac{dy}{dx} = x - y^2$, y(0)=1 and compute y(0.1) correct to four decimal places.

Sol: Consider $\frac{dy}{dx} = f(x, y)$ and $y(x_0)=y_0$.

Here $f(x,y)=x-y^2$, $y_0=1$ and $x_0=0$.

By picard's method, a sequence of successive approximation to y are given by

$$y^{(n)}(x) = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})dx \text{ , n=1,2,3,---}$$

$$y^{(n)}(x) = 1 + \int_{0}^{x} f(x, y^{(n-1)})dx \text{ , n=1,2,3,---}$$

First approximation: we have

$$y^{(1)}(x) = 1 + \int_0^x f(x, y^0)dx = 1 + \int_0^x f(x,1)dx = 1 + \int_0^x (x-1)dx = 1 + \frac{x^2}{2} - x$$

Second approximation, we have

$$y^{(2)}(x) = 1 + \int_0^x f(x, y^1)dx = 1 + \int_0^x \left[x - \left(1 + \frac{x^2}{2} - x\right)^2 \right] dx$$

$$= 1 + \int_0^x \left[(x - (1 + \frac{x^4}{4} + x^2 + x^2 - 2x - x^3) \right] dx$$

$$= 1 + \int_0^x (3x - 1 - \frac{x^4}{4} - 2x^2 + x^3)dx = 1 + \frac{3x^2}{2} - x - \frac{x^5}{20} - \frac{2x^3}{3} + \frac{x^4}{4}$$

It is clear that the resulting expressions too big as we proceed to higher approximations. Hence approximate value of y(x) is $y^{(2)}(x)$.

For x=0.1

$$y^{(2)} = 1 - 0.1 + \frac{3}{2}(0.1)^2 - \frac{2}{3}(0.1)^3 + \frac{1}{4}(0.1)^4 - \frac{(0.1)^5}{20}$$

= 1-0.1+0.015-0.0006666+0.000025-0.0000005

= 1.015025-0.1006671

= 0.9143579 ~ 0.9143 (correct to four decimal places)

y = 0.9143 at x=0.1.

7. Find the value of y for x=0.25, 0.5, 1 by Picard's method, given that $\frac{dy}{dx} = \frac{x^2}{y^2+1}$ **and** $x_0=0$**,** $y_0 = 0$**.**

Sol: Consider $\frac{dy}{dx} = f(x, y)$ and $y(x_0)=y_0$ or $y=y_0$ at $x=x_0$

Here $f(x, y) = \frac{x^2}{y^2+1}$ and $x_0=0$, $y_0=0$

By picard's method a sequence of approximations are given by

$$y^{(n)}(x) = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})dx \text{, n=1,2,3,---}$$

$$y^{(n)}(x) = 0 + \int_{0}^{x} f(x, y^{(n-1)})dx \text{, n=1,2,3--- (1)}$$

First approximation: we have

$$y^{(1)}(x) = 0 + \int_0^x f(x, y^0)dx = 0 + \int_0^x f(x,1)dx = 0 + \int_0^x \frac{x^2}{0^2+1}dx = 0 + \frac{x^3}{3}$$

Second approximation, we have

$$y^{(2)}(x) = 0 + \int_0^x f(x, y^{(1)})dx = \int_0^x \frac{x^2}{(y^{(1)})^2+1}dx = \int_0^x \frac{x^2}{(\frac{x^3}{3})^2+1}dx$$

$$= \tan^{-1}(\frac{x^3}{3}) - 0 \text{ [by putting } \frac{x^3}{3}\text{=t] } = \tan^{-1}(\frac{x^3}{3})$$

Third approximation, we have

$$y^{(3)}(x) = \int_0^x f(x, y^{(2)})dx = \int_0^x \frac{x^2}{[\tan^{-1}(\frac{x^3}{3})]^2+1}dx$$

The integration is difficult, this is the drawback of the method. Hence the approximation value of y is $y^{(2)}(x)$.

$$y^{(2)}(x) = \tan^{-1}(\frac{x^3}{3}) = \frac{x^3}{3} - (\frac{x^3}{3})^3\frac{1}{3} + (\frac{x^3}{3})^5\frac{1}{5} - - -$$

$$= \frac{x^3}{3} - \frac{x^9}{81} + \frac{x^{15}}{1215} - - - - (\tan^{-1}(2) = x - \frac{x^3}{3} + \frac{x^5}{5} - - -)$$

For x=0.25,

$$y^{(2)}(x) = \frac{(0.25)^3}{3} - \frac{(0.25)^9}{81} + \frac{(0.25)^{15}}{1215} = 0.0052082$$

at x=0.5,

$$y^{(2)}(x) = \frac{(0.5)^3}{3} - \frac{(0.5)^9}{81} + \frac{(0.5)^{15}}{1215} = 0.0416425$$

At x=1,

$$y^{(2)}(x) = \frac{1}{3} - \frac{1}{81} - \frac{1}{1215} = 0.32180699$$

y=0.0052082 at x=0.25

y=0.0416425 at x=0.5

y=0.32180699 at x=1

8. Given $\frac{dy}{dx} = xe^y$, y(0)=0, determine y(0.1), y(0.2) and y(1) using picard's method.

Sol: Consider $\frac{dy}{dx} = f(x,y)$ and $y(x_0)=y_0$

Here f(x,y)= xe^y, $x_0=0$ and $y_0=0$

By picard's method, a sequence of approximations are given by

$$y^{(n)}(x) = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})dx, \text{ n=1,2,3,---}$$

$$\therefore y^{(n)}(x) = \int_{0}^{x} f(x, y^{(n-1)})dx, \text{ n=1, 2, 3,---- (1)}$$

First approximation, we have

$$y^{(1)}(x) = \int_0^x f(x, y^0)dx = \int_0^x x.e^0 dx = \frac{x^2}{2}$$

Second approximation, we have

$$y^{(2)}(x)=\int_0^x f(x,y^{(1)})dx=\int_0^x x.e^{\frac{x^2}{2}}dx=e^{\frac{x^2}{2}}-1$$

Third approximation, we have

$$y^{(3)}(x)=\int_0^x f(x,y^{(2)})dx=\int_0^x x.e^{(x)\frac{x^2}{2}}dx$$

The integration is difficult, Hence the approximate value of y is $y^{(2)}(x)$.

$$y^{(2)}(x)=e^{\frac{x^2}{2}}-1$$

for x=0.1, $y^{(2)}(x)=e^{\frac{(0.1)^2}{2}}-1=0.005012$

for x=0.2, $y^{(2)}(x)=e^{\frac{(0.2)^2}{2}}-1=0.02020$

For x=1, $y^{(2)}(1)=e^{\frac{1}{2}}$

TAYLOR'S SERIES METHOD

To find the numerical solution of the differential equation $\frac{dy}{dx}=f(x,y)\rightarrow(1)$

With the initial condition $y(x_0)=y_0\rightarrow(2)$

$y(x)$ Can be expanded about the point x_0 in a Taylor's series in powers of $(x-x_0)$ as

$$y(x)=y(x_0)+\frac{(x-x_0)}{1!}y^{I}(x_0)+\frac{(x-x_0)^2}{2!}y^{II}(x_0)+.+\frac{(x-x_0)^n}{n!}y^{n}(x_0)+..\rightarrow(3)$$

In equation (3), $y(x_0)$ is known from initial condition equation. The remaining coefficients $y'(x_0), y''(x_0),.........y^n(x_0)$ etc are obtained by successively differentiating equation (1) and evaluating at x_0. Substituting these values in equation, $y(x)$ at any point can be calculated from equation. Provided $h=x-x_0$ is small.

When $x_0=0$, then Taylor's series equation can be written as

$$y(x)=y(0)+x.y'(0)+\frac{x^2}{2!}y''(0)+......+\frac{x^n}{n!}y^n(0)+........\rightarrow(4)$$

Note: We know that the Taylor's expansion of y(x) about the point x_0 in a power of $(x-x_0)$ is.

$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \ldots \rightarrow (1)$ Or

$y(x) = y_0 + \frac{(x-x_0)}{1!} y_0^I + \frac{(x-x_0)^2}{2!} y_0^{II} + \frac{(x-x_0)^3}{3!} y_0^{III} + \ldots\ldots$

If we let $x - x_0 = h$. (i.e. $x = x_0 + h = x_1$) we can write the Taylor's series as

$$y(x) = y(x_1) = y_0 + \frac{h}{1!} y_0^I + \frac{h^2}{2!} y_0^{II} + \frac{h^3}{3!} y_0^{III} + \frac{h^4}{4!} y_0^{IV} + \ldots.$$

$$\text{i.e. } y_1 = y_0 + \frac{h}{1!} y_0^I + \frac{h^2}{2!} y_0^{II} + \frac{h^3}{3!} y_0^{III} + \frac{h^{IV}}{4!} y_0^{IV} + \ldots\ldots \qquad \rightarrow (2)$$

Similarly expanding y(x) in a Taylor's series about $x = x_1$. We will get.

$$y_2 = y_1 + \frac{h}{1!} y_1^I + \frac{h^2}{2!} y_1^{II} + \frac{h^3}{3!} y_1^{III} + \frac{h^4}{4!} y_1^{IV} + \ldots\ldots. \qquad \rightarrow (3)$$

Similarly expanding y(x) in a Taylor's series about $x = x_2$ We will get.

$$y_3 = y_2 + \frac{h}{1!} y_2^I + \frac{h^2}{2!} y_2^{II} + \frac{h^3}{3!} y_2^{III} + \frac{h^4}{4!} y_2^{IV} + \ldots\ldots \qquad \rightarrow (4)$$

In general, Taylor's expansion of y(x) at a point $x = x_n$ is

$$y_{n+1} = y_n + \frac{h}{1!} y_n^I + \frac{h^2}{2!} y_n^{II} + \frac{h^3}{3!} y_n^{III} + \frac{h^4}{4!} y_n^{IV} + \ldots. \ldots\ldots\ldots\ldots.. \qquad (I)$$

Merits and Demerits of Taylor series method:

In this method taking h very small and taking upto order h^4 terms we get less error but finding derivatives may be complicate in some of the problems

PROBLEMS:

1. **Solve $\frac{dy}{dx} = xy + 1$ and y(0) = 1 using Taylor's series method and compute y(0.1).**

SOL:. Given that $\frac{dy}{dx}$ - 1 = xy and y(0) = 1

Here $\frac{dy}{dx} = 1 + xy$ and $y_0 = 1$, $x_0 = 0$.

Differentiating repeatedly w.r.t 'x' and evaluating at $x_0 = 0$

$y^I(x) = 1 + xy$, $\qquad y^I(0) = 1+0(1) = 1$.

$y^{II}(x) = x.y^I + y$, $\qquad y^{II}(0) = 0+1=1$

$y^{III}(x) = x.y^{II} + y^{I} + y^{I}$ $y^{III}(0) = 0.(1) + 2(1) = 2$

The Taylor series for f(x) about $x_0 = 0$ is

$y(x) = y(0) + x.y^{'}(0) + \frac{x^2}{2!} y^{''}(0) + \frac{x^3}{3!} y^{'''}(0)$ (Neglecting higher order terms)

Substituting the values of $y(0)$, $y^{'}(0)$, $y^{''}(0)$, ….

$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}(2)$

$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3}$ →(1)

Now put x = 0.1 in equ (1),

$y(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3}.$

= 1 + 0.1 +0.005 + 0.000333 = **1.105**

2. Solve the equation $\frac{dy}{dx} = x - y^2$ with the conditions y(0) = 1 and y`(0) =1. Find y(0.2) and y(0.4) using Taylor's series method.

SOL: Given that $y' = x - y^2, y(0) = 1$ Here $y_0 = 1$, $x_0 = 0$

Differentiating repeatedly w.r.t 'x' and evaluating at x=0

$$y'(x) = x - y^2, y'(0) = 0 - y(0)^2 = 0 - 1 = -1$$
$$y''(x) = 1 - 2y.y', y''(0) = 1 - 2.y(0)y'(0) = 1 - 2(-1) = 3$$
$$y'''(x) = 1 - 2yy'' - 2(y')^2, y'''(0) = -2.y(0).y''(0) - 2.(y'(0))^2 = -6 - 2 = -8$$

The Taylor's series for f(x) about $x_0 = 0$ is

$y(x) = y(0) + \frac{x}{1!} y^{1}(0) + \frac{x^2}{2!} y^{11}(0) + \frac{x^3}{3!} y^{111}(0)$ (Neglecting higher order terms)

Substituting the value of $y(0)$, $y^{1}(0)$, $y^{11}(0)$,…..

$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{8}{6}x^3$

$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3$ →(1)

Now put x = 0.1 in (1)

$y(0.1) = 1 - 0.1 + \frac{3}{2}(0.1)^2 + \frac{4}{3}(0.1)^3$ = **0.9138**

Similarly put x = 0.2 in (1)

$y(0.2) = 1 - 0.2 + \frac{3}{2}(0.2)^2 - \frac{4}{3}(0.2)^3$ = **0.8516.**

3. Tabulate y(0.1), y(0.2) and y(0.3) using Taylor's series method given that $y^1 = y^2 + x$ and y(0) = 1.

Sol: Given $y^1 = y^2 + x$ →(1) and $y(0) = 1$ →(2)

Here $x_0 = 0$, $y_0 = 1$.Take h=0.1then x_1=x0+h=0.1;x_2=0.2;x_3=0.3

Differentiating (1) w.r.t 'x', we get

$y^{II} = 2y \cdot y^I + 1$ →(3)

$y^{III} = 2[y \cdot y^{II} + (y^I)^2]$ →(4)

$y^{IV} = 2[y \cdot y^{III} + y^I y^{II} + 2 y^I y^{II}] = 2[y \cdot y^{III} + 3 y^I y^{II}]$ →(5)

Put $x_0 = 0$, $y_0 = 1$ in (1), (3), (4) and (5), we get

$y_0^I = (1)^2 + 0 = 1$

$y_0^{II} = 2(1)(1) + 1 = 3,$

$y_0^{III} = 2((1)(3) + (1)^2) =$

$y_0^{IV} = 2[(1)(8) + 3(1)(3)] = 34$

Take h = 0.1.

__Step1__: By Taylor's series expansion, we have

$y(x_1) = y_1 = y_0 + \frac{h}{1!}y_0^I + \frac{h^2}{2!}y_0^{II} + \frac{h^3}{3!}y_0^{III} + \frac{h^4}{4!}y_0^{IV} + \ldots$ →(6)

on substituting the values of y_o, y_0^I, y_0^{II} etc in (6),we get

$y(0.1) = y_1 = 1 + (0.1)(1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(8) + \frac{(0.1)^4}{24}(34) + \ldots$

$= 1 + 0.1 + 0.015 + 0.001333 + 0.000416 \Rightarrow y_1 = 1.116749$

Step2: Let us find y(0.2),we start with (x_1,y_1) as the starting values

Here $x_1 = x_0 + h = 0 + 0.1 = 0.1$ and $y_1 = 1.116749$

Putting these values in (1),(3),(4) and (5), we get

$y_1^I = y_1^2 + x_1 = (1.116749)^2 + 0.1 = 1.3471283$

$y_1^{II} = 2y_1\, y_1^I + 1 = 2(10116749)\,(1.3471283) + 1 = 4.0088$

$y_1^{III} = 2(y_1\, y_1^{II} + (y_1^I)^2) = 2((1.116749)\,(4.0088) + (1.3471283)^2] = 12.5831$

$y_1^{IV} = 2y_1\, y_1^{III} + 6\, y_1^I\, y_1^{II} = 2(1.116749)\,(12.5831) + 6(1.3471283)\,(4.0088) = 60.50653$

By Taylor's expansion

$$y(x_2) = y_2 = y_1 + \frac{h}{1!} y_1^I + \frac{h^2}{2!} y_1^{II} + \frac{h^3}{3!} y_1^{III} + \frac{h^4}{4!} y_1^{IV} + \ldots.$$

$$y(0.2)=y_2=1.116749 + (0.1)(1.3471283)+\frac{(0.1)^2}{2}(4.0088)+\frac{(0.1)^3}{6}(12.5831)+\frac{(0.1)^4}{24}(60.50653)$$

$y_2 = 1.116749 + 0.13471283 + 0.020044 + 0.002097 + 0.000252 = 1.27385$

$y(0.2) = 1.27385$

Step3: Let us find y(0.3),we start with (x_2,y_2) as the starting value.

Here $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$ and $y_2 = 1.27385$

Putting these values of x_2 and y_2 in eq (1), (3), (4) and (5), we get

$y_2^I = y_2^2 + x_2 = (1.27385)^2 + 0.2 = 1.82269$

$y_2^{II} = 2y_2\, y_2^1 + 1 = 2(1.27385)\,(1.82269) + 1 = 5.64366$

$y_2^{III} = 2[y_2\, y_2^{II} + (y_2^I)^2] = 2[(1.27385)\,(5.64366) + (1.82269)^2]$

$= 14.37835 + 6.64439 = 21.02274$

$y_2^{IV} = 2y_2 + y_2^{III} + 6\, y_2^I \cdot y_1^{II} = 2(1.27385)\,(21.00274) + 6(1.82269)\,(5.64366)$

$= 53.559635 + 61.719856 = 115.27949$

By Taylor's expansion,

$$y(x_3) = y_3 = y_2 + \frac{h}{1!} y_2^I + \frac{h^2}{2!} y_2^{II} + \frac{h^3}{3!} y_2^{III} + \frac{h^4}{4!} y_2^{IV} + \ldots..$$

$y(0.3)=y_3=1.27385+(0.1)(1.82269)+\frac{(0.1)^2}{2}(5.64366)+\frac{(0.1)^3}{6}(21.02274)+\frac{(0.1)^4}{24}(115.27949)$

$= 1.27385 + 0.182269 + 0.02821 + 0.0035037 + 0.00048033 = 1.48831$

$y(0.3) = 1.48831$

4. Solve $y^1 = x^2 - y$, $y(0) = 1$ using Taylor's series method and evaluate y(0.1),y(0.2),y(0.3) and y(0.4) (correct to 4 decimal places)

Sol: Given $y^1 = x^2 - y$ →(1) and $y(0) = 1$ →(2)

Here $x_0 = 0$, $y_0 = 1$

Differentiating (1) w.r.t 'x', we get

$y^{II} = 2x - y^1$ →(3)

$y^{III} = 2- y^{II}$ →(4)

$y^{IV} = -y^{III}$ →(5)

put $x_0 = 0$, $y_0 = 1$ in (1),(3),(4) and (5), we get

$y_0^I = x_0^2 - y_0 = 0-1 = -1$,

$y_0^{II} = 2x_0 - y_0^I = 2(0) - (-1) = 1$

$y_0^{III} = 2- y_0^{II} = 2-1 = 1$,

$y_0^{IV} = - y_0^{III} = -1$ Take $h = 0.1$

Step1: by Taylor's series expansion

$y(x_1) = y_1= y_0 + \frac{h}{1!}y_0^I + \frac{h^2}{2!}y_0^{II} + \frac{h^3}{3!}y_0^{III} + \frac{h^4}{4!}y_0^{IV} + \ldots$ →(6)

On substituting the values of y_0, y_0^I, y_0^{II} etc in (6), we get

$y(0.1) = y_1 = 1+ (0.1)(-1) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(1) + \frac{(0.1)^4}{24}(-1)+\ldots$

$= 1-0.1 + 0.005 + 0.01666 - 0.0000416+\ldots$

$= 0.905125 \simeq 0.9051$ (4 decimal place).

Step2: Let us find y(0.2) we start with (x_1,y_1) as the starting values

Here $x = x_0 + h = 0 + 0.1 = 0.1$ and $y_1 = 0.905125$,

Putting these values of x_1 and y_1 in (1), (3), (4) and (5), we get

$y_1^1 = x_1^2 - y_1 = (0.1)^2 - 0.905125 = -0.895125$

$y_1^{II} = 2x_1 - y_1^1 = 2(0.1) - (-0.895125) = 1.095125,$

$y_1^{III} = 2 - y_1^{II} = 2 - 1.095125 = 0.904875,$

$y_1^{IV} = - y_1^{III} = -0.904875,$

By Taylor's series expansion,

$$y(x_2) = y_2 = y_1 + \frac{h}{1!} y_1^I + \frac{h^2}{2!} y_1^{II} + \frac{h^3}{3!} y_1^{III} + \frac{h^4}{4!} y_1^{IV} + \ldots$$

$$y(0.2)=y_2=0.905125+(0.1)(-0.895125)+\frac{(0.1)^2}{2}(1.09125)+\frac{(0.1)^3}{6}(0.904875)+\frac{(0.1)^4}{24}(-0.904875)+\ldots$$

$y(0.2) = y_2 = 0.905125 - 0.0895125 + 0.00547562 + 0.000150812 - 0.00000377$

$= 0.8212351 \simeq 0.8212$ (4 decimal places)

Step3: Let us find y(0.3), we start with (x_2,y_2) as the starting value

Here $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$ and $y_2 = 0.8212351$

Putting these values of x_2 and y_2 in (1),(3),(4), and (5) we get

$y_2^1 = x_2^2 - y_2 = (0.2)^2 - 0.8212351 = 0.04 - 0.8212351 = - 0.7812351$

$y_2^{II} = 2x_2 - y_2^1 = 2(0.2) + (0.7812351) = 1.1812351,$

$y_2^{III} = 2 - y_2^{II} = 2 - 1. 1812351 = 0.818765,$

$y_2^{IV} = - y_2^{III} = -0.818765,$

By Taylor's series expansion,

$$y(x_3) = y_3 = y_2 + \frac{h}{1!} y_2^I + \frac{h^2}{2!} y_2^{II} + \frac{h^3}{3!} y_2^{III} + \frac{h^4}{4!} y_2^{IV} + \ldots$$

$$y(0.3)=y_3=0.8212351+(0.1)(-0.7812351)+\frac{(0.1)^2}{2}(1.1812351)+\frac{(0.1)^3}{6}(0.818765)+\frac{(0.1)^4}{24}(-0.818765)+\ldots$$

$y(0.3) = y_3 = 0.8212351 - 0.07812351 + 0.005906 + 0.000136 - 0.0000034$

$= 0.749150 \simeq 0.7492$ (4 decimal places)

Step4: Let us find y(0.4), we start with (x_3,y_3) as the starting value

Here $x_3 = x_2 + h = 0.2+ 0.1 = 0.3$ and $y_3 = 0.749150$

Putting these values of x_3 and y_3 in (1),(3),(4), and (5) we get

$y_3^1 = x_3^2 - y_3 = (0.3)^2 - 0.749150 = -0.65915,$

$y_3^{II} = 2x_3 - y_3^1 = 2(0.3) + (0.65915) = 1.25915,$

$y_3^{III} = 2 - y_3^{II} = 2 - 1.25915 = 0.74085,$

$y_3^{IV} = - y_3^{III} = -0.74085,$

By Taylor's series expansion,

$$y(x_4) = y_4 = y_3 + \frac{h}{1!} y_3^I + \frac{h^2}{2!} y_3^{II} + \frac{h^3}{3!} y_3^{III} + \frac{h^4}{4!} y_3^{IV} + \ldots$$

$$y(0.4) = y_4 = .749150+(.1)(-.65915)+\frac{(0.1)^2}{2}(1.25915)+\frac{(0.1)^3}{6}(.74085)+\frac{(0.1)^4}{24}(-.74085)+\ldots$$

$y(0.4) = y_4 = 0.749150 - 0.065915 + 0.0062926 + 0.000123475 - 0.0000030$

$= 0.6896514 \simeq 0.6897$ (4 decimal places)

5. Using Taylor's expansion evaluate the integral of $y' - 2y = 3e^x, y(0) = 0$**,** at

a) $x = 0.1, 0.2, 0.3,$ b) Compare the numerical solution obtained with exact solution.

Sol: Given equation can be written as $2y + 3e^x = y', y(0) = 0$

Differentiating repeatedly w.r.t to'x' and evaluating at $x = 0$

$$y'(x) = 2y + 3e^x, y'(0) = 2y(0) + 3e^0 = 2(0) + 3(1) = 3$$
$$y''(x) = 2y' + 3e^x, y''(0) = 2y'(0) + 3e^0 = 2(3) + 3 = 9$$
$$y'''(x) = 2.y''(x) + 3e^x, y'''(0) = 2y''(0) + 3e^0 = 2(9) + 3 = 21$$
$$y^{iv}(x) = 2.y'''(x) + 3e^x, y^{iv}(0) = 2(21) + 3e^0 = 45$$
$$y^{v}(x) = 2.y^{iv} + 3e^x, y^{v}(0) = 2(45) + 3e^0 = 90 + 3 = 93$$

In general, $y^{(n+1)}(x) = 2.y^{(n)}(x) + 3e^x$ or $y^{(n+1)}(0) = 2.y^{(n)}(0) + 3e^0$

The Taylor's series expansion of $y(x)$ about $x_0 = 0$ is

$$y(x) = y(0) + xy^1(0) + \frac{x^2}{2!}y^{11}(0) + \frac{x^3}{3!}y^{111}(0) + \frac{x^4}{4!}y^{1v}(0) + \frac{x^5}{5!}y^V(0) + \cdots$$

Substituting the values of $y(0), y'(0), y''(0), y'''(0),$..........

$$y(x)=0+3x+\frac{9}{2}x^2+\frac{21}{6}x^3+\frac{45}{24}x^4+\frac{93}{120}x^5+........$$

$$y(x)=3x+\frac{9}{2}x^2+\frac{7}{2}x^3+\frac{15}{8}x^4+\frac{31}{40}x^5+........\rightarrow (1)$$

Now put $x=0.1$ in equation

$$y(0.1)=3(0.1)+\frac{9}{2}(0.1)^2+\frac{7}{2}(0.1)^3+\frac{15}{8}(0.1)^4+\frac{31}{40}(0.1)^5=0.34869$$

Now put $x=0.2$ in equation

$$y(0.2)=3(0.2)+\frac{9}{2}(0.2)^2+\frac{7}{2}(0.2)^3+\frac{15}{8}(0.2)^4+\frac{31}{40}(0.2)^5=0.811244$$

Now put $x = 0.3\ in\ equation(1)$

$$y(0.3)=3(0.3)+\frac{9}{2}(0.3)^2+\frac{7}{2}(0.3)^3+\frac{15}{8}(0.3)^4+\frac{31}{40}(0.3)^5=1.41657075$$

Analytical Solution:

The exact solution of the equation $\frac{dy}{dx}=2y+3e^x$ with $y(0)=0$ can be found as follows

$\frac{dy}{dx}-2y=3e^x$ This is a linear in y.

Here $P=-2, Q=3e^x$

I.F $=e^{\int p(x)dx} = e^{\int -2xdx} = e^{-2x}$

General solution is $y.e^{-2x}=\int 3e^x.e^{-2x}dx+c=-3e^{-x}+c$

$\therefore y=-3e^x+ce^{2x}$ Where $x=0, y=0 \quad 0=-3+c \Rightarrow c=3$

The particular solution is $y=3e^{2x}-3e^x$ or $y(x)=3e^{2x}-3e^x$

Put $x=0.1$ in the above particular solution,

$$y=3.e^{0.2}-3e^{0.1}=0.34869$$

Similarly put $x=0.2$

$$y=3e^{0.4}-3e^{0.2}=0.811265$$

put $x=0.3$ $\quad y=3e^{0.6}-3e^{0.3}=1.416577$

6. Using Taylor's series method, solve the equation $\frac{dy}{dx} = x^2 + y^2$ **for** $x = 0.4$ **given that y = 0 when x = 0**

Sol: Given equation is $\frac{dy}{dx} = x^2 + y^2$ and $y = 0$ when $x = 0$ i.e. $y(0) = 0$

Here $y_0 = 0$, $x_0 = 0$

Differentiating repeatedly w.r.t 'x' and evaluating at $x = 0$

$$y^1(x) = x^2 + y^2, y^1(0) = 0 + y^2(0) = 0 + 0 = 0$$

$$y^{11}(x) = 2x + y^I.2y, y^{11}(0) = 2(0) + y^1(0)2.y = 0$$

$$y^{111}(x) = 2 + 2yy^{11} + 2y^1.y^1, y^{111}(0) = 2 + 2.y(0).y^1(0) + 2.y^1(0)^2 = 2$$

$$y^{IV}(x) = 2.yy^{111} + 2.y^{11}, y^1 + 4.y^{11}.y^1, y^1(0) = 0$$

The Taylor's series for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y''''(0) + ...$$

Substituting the values of $y(0), y'(0), y''(0), ...$

$$y(x) = 0 + x(0) + 0 + \frac{2x^3}{3!} + 0 + ... = \frac{x^3}{3} + \text{(Higher order terms are neglected)}$$

$$\therefore y(0.4) = \frac{(0.4)^3}{3} = \frac{0.064}{3} = 0.02133$$

7. Find y (0.1),y (0.2), z(0.1), z(0.2) given $\frac{dy}{dx} = x + y, \frac{dz}{dx} = x - y^2$ *and* $y(0) = 2, z(0) = 1$ **by Using Taylor's series method**

SOL: Given y^I =x+z, take $x_0 = 0$, $y_0 = 2$,h=0.1

We have to find $y_1 = y(0.1)$ and $y_2 = y(0.2)$

Now y^I = x+z, y^{11}= 1+z^1, y^{111}= z^{11} ....(I)

Given z^1 =x-y^2

take $x_0 = 0$, $z_0 = 1$,h=0.1

we have to find $z_1 = z(0.1)$ and $z_2 = z(0.2)$

now z^1 =x-y^2, z^{11} =1-2y.y^1, z^{111}=-2[y.y^{11} + $(y^1)^2$ ....(II)

By Taylor's series ,for y_1 and z_1 ,we have

$y(x) = y_0 + h.y^I_0 + \frac{h^2}{2!} y_0^{II} + \frac{h^3}{3!} y_0^{III}$ (neglecting higher order terms)….(1)

$z(x) = z_0 + h.z^I_0 + \frac{h^2}{2!} z_0^{II} + \frac{h^3}{3!} z_0^{III}$ (neglecting higher order terms)……………….(2)

From (I) and (II), we get

$y_0 = 2$; $z_0 = 1$;

$y_0^I = x_0 + z_0 = 0+1 = 1$ $z_0^1 = x_0 - y_0^2 = -4$

$y^{II}_0 = 1 + z_0^1 = 1 + x_0 - y_0^2 = 1+0-4 = -3$; $z_0^{11} = 1 - 2y_0.y_0^1 = 1-2(2)1 = -3$

$y_0^{111} = z_0^{11} = -3$; $z_0^{111} == -2[y_0.y_0^{11} + (y_0)^2 = 10$

Substituting these values in (1) and(2)

$y_1 = y(0.1) =$ $2+(0.1)1+\frac{0.01}{2}(-3)+\frac{0.001}{6}(-3)=2.0845.$

$z_1 = z(0.1) = 1+(0.1)(-4)+\frac{0.01}{2}(-3)+\frac{0.001}{6}(10)=0.5867.$

Similarly

By Taylor's series for y_2, z_2 are

$y_2 = y_1 + h.y^I_1 + \frac{h^2}{2!} y_1^{II} + \frac{h^3}{3!} y_1^{III}$………………..(3)

$z_2 = z_1 + h.z^I_1 + \frac{h^2}{2!} z_1^{II} + \frac{h^3}{3!} z_1^{III}$…………………..(4)

Now we have

$y_1 = 2.01845$; $z_1 = 1$;

$y_1^I = x_1 + z_1 = 0.1+0.5867 = 0.6867$ $z_1^1 = x_1 - y_1^2 = -4.2451$

$y_1^{II} = 1 + z_1^1 = 1 + x_1 - y_1^2 = -3.2451$; $z_1^{11} = 1 - 2y_1.y_1^1 = -1.8628$

$y_1^{111} = z_1^{11} = -1.8628$; $z_1^{111} == -2[y_1.y_1^{11} + (y_1)^2 = 12.5856$

Substituting in (3) and (4).We get

$y_2 = y(0.2) =$ $2.0845+(0.1)(0.6867)+\frac{0.01}{2}(-3.2451)+\frac{0.001}{6}(-1.8628)=2.1367.$

$z_2 = z(0.2) = 0.5867+(0.1)(-4.2451)+\frac{0.01}{2}(-1.8628)+\frac{0.001}{6}(12.5856)=0.15497.$

EULER'S METHOD:-

It is the simplest one-step method and it is less accurate. Hence it has a limited application.

Consider the differential equation $\frac{dy}{dx} = f(x,y)$ →(1)   With $y(x_0) = y_0$ →(2)

Consider the first two terms of the Taylor's expansion of y(x) at $x = x_0$

$y(x) = y(x_0) + (x - x_0)\, y^1(x_0)$ →(3)

from equation (1) $y^1(x_0) = f(x_0, y(x_0)) = f(x_0, y_0)$

Substituting in equation (3)

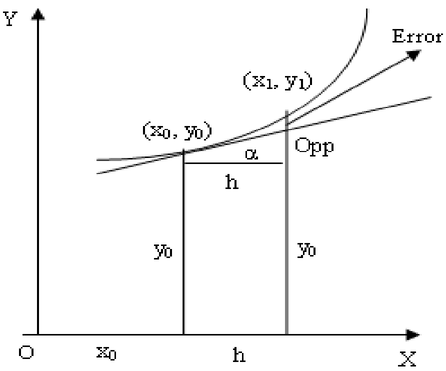
$\therefore y(x) = y(x_0) + (x - x_0)\, f(x_0, y_0)$ At $x = x_1$, $y(x_1) = y(x_0) + (x_1 - x_0)\, f(x_0, y_0)$

$\therefore y_1 = y_0 + h\, f(x_0, y_0)$ where $h = x_1 - x_0$

Similarly at $x = x_2$, $y_2 = y_1 + h\, f(x_1, y_1)$

Proceeding as above, $y_{n+1} = y_n + h\, f(x_n, y_n)$

This is known as Euler's Method



From the fig,

$\text{Tan}\alpha = \frac{opp}{adj} = \frac{opp}{h}$

Implies opp = h $\text{Tan}\alpha$

But $\text{Tan}\alpha$=slope at (x_0, y_0) $= \frac{dy}{dx}\, at (x_0, y_0) = f(x_0, y_0)$

$\therefore$opp=h $f(x_0, y_0)$

Hence $y_1 = y_0 + opp$ implies $y_1 = y_0 + h\, f(x_0, y_0)$ [NEGLECTING ERROR]

We remove that error by using EULER'S MODIFIED METHOD.

PROBLEMS:

1. Using Euler's method, solve for y at x = 2 from $\frac{dy}{dx} = 3x^2 + 1, y(1) = 2$,taking step size (i)

h = 0.5 and (ii) h=0.25

Sol: Here $f(x,y) = 3x^2 + 1$, $x_0 = 1, y_0 = 2$

Euler's algorithm is $y_{n+1} = y_n + h\ f(x_n,y_n)$, n = 0,1,2,3,….. →(1)

(i) h = 0.5 $\therefore x_1 = x_0 + h = 1+0.5 = 1.5$

Taking n = 0 in (1) , we have $x_2 = x_1 + h = 1.5 + 0.5 = 2$

$y_1 = y_0 + h\ f(x_0,y_0)$

i.e. $y_1 = y(1.5) = 2 + (0.5)\ f(1,2) = 2 + (0.5)\ (3 + 1) = 2 + (0.5)(4) = 4$

Here $x_1 = x_0 + h = 1 + 0.5 = 1.5$

$\therefore y(1.5) = 4 = y_1$

Taking n = 1 in (1),we have

$y_2 = y_1 + h\ f(x_1,y_1)$

i.e. $y(x_2) = y_2 = 4 + (0.5)\ f(1.5,4) = 4 + (0.5)[3(1.5)^2 + 1] = 7.875$

Here $x_2 = x_4 + h = 1.5 + 0.5 = 2$

$\therefore y(2) = 7.875$

(ii) h = 0.25 $\therefore x_1 = 1.25,\ x_2 = 1.50,\ x_3 = 1.75,\ x_4 = 2$

Taking n = 0 in (1), we have

$y_1 = y_0 + h\ f(x_0,y_0)$

i.e. $y(x_1) = y_1 = 2 + (0.25)\ f(1,2) = 2 + (0.25)\ (3 + 1) = 3$

$y(x_2) = y_2 = y_1 + h\ f(x_1,y_1)$

i.e. $y(x_2) = y_2 = 3 + (0.25)\ f(1.25,3) = 3 + (0.25)[3(1.25)^2 + 1] = 5.42188$

Here $x_2 = x_1 + h = 1.25 + 0.25 = 1.5$

$\therefore y(1.5) = 5.42188$

Taking n = 2 in (1), we have

i.e. $y(x_3) = y_3 = y_2 + h\ f(x_2,y_2)$

$= 5.42188 + (0.25)\ f(1.5,5.42188)$

$= 5.42188 + (0.25)\ [3(1.5)^2 + 1] = 7.35938$

Here $x_3 = x_2 + h = 1.5 + 0.25 = 1.75$

$\therefore y(1.75) = 7.35938$

Taking n = 4 in (1),we have

$y(x_4) = y_4 = y_3 + h\, f(x_3, y_3)$

i.e. $y(x_4) = y_4 = 7.35938 + (0.25)\, f(1.75, 7.35938)$

$$= 7.35938 + (0.25)[3(1.75)^2 + 1] = 9.90626$$

Note that the difference in values of y(2) in both cases (i.e. when h = 0.5 and when h = 0.25).The accuracy is improved significantly when h is reduced to 0.25 (Exact solution of the equation is $y = x^3 + x$ and with this $y(2) = y_2 = 10$.

2. Solve by Euler's method,$y^1 = x + y$, $y(0) = 1$ and find y(0.3) taking step size h = 0.1. compare the result obtained by this method with the result obtained by analytical solution

Sol: Here $f(x,y) = x+y$, $x_0 = 0, y_0 = 1$

Euler's algorithm is $y_{n+1} = y_n + h\ f(x_n, y_n)$, n = 0,1,2,3,….. →(1)

Given h = 0.5 $\therefore x_1 = x_0 + h = 0+0.1 = 0.1$

Taking n = 0 in (1) , we have $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

$y_1 = y_0 + h\, f(x_0, y_0)$

i.e. $y_1 = y(0.1) = 1 + (0.1)\, f(0,1) = 1.1$

$\therefore y(0.1) = 1.1$

Here $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

Taking n = 1 in (1),we have $y_2 = y_1 + h\, f(x_1, y_1)$

i.e. $y(x_2) = y_2 = 1.1 + (0.1)\, f(0.1.1.1) = 1.22$

Similarly we get $y_3 = y(0.3) = 1.362$

Analytical solution:

The exact solution of $\frac{dy}{dx} = x + y$, y(0)=1 can be found as follows.

The equation can be written as $\frac{dy}{dx} - y = x$

This is a linear equation in y [i.e, $\frac{dy}{dx} + p.y = Q$]

then p=-1, Q=x. $I.F = e^{\int pdx} = e^{\int (-1)dx} = e^{-x}$

General solution is y . I.F= $\int QXI.Fdx + c$

$$y.e^{-x} = \int x.e^{-x} dx + c$$

$$y.e^{-x} = - e^{-x}(x+1)+c. \text{ or } y=-(x+1)+ce^{+x}$$

when x=0, y=1 i.e, i=-(0+1)+c or c=2

Hence the particular solution of the equation is

$y=-(x+1)+2e^x=2e^x-x-1$.

Particular solution is $y = 2e^x – (x + 1)$

Hence y(0.1) = 1.11034, y(0.2) = 1.3428, y(0.3) = 1.5997

We shall tabulate the result as follows

X	0	0.1	0.2	0.3
Euler y	1	1.1	1.22	1.362
Exact y	1	1.11034	1.3428	1.5997

The value of y deviate from the exact value as x increases. This indicate that the method is not accurate

3. Given $y' = x^2 - y, y(0) = 1$ **find correct to four decimal places the value of y (0,1), by using Euler's method.**

Sol: We have $f(x, y) = x^2 - y$ $x_0 = 0; y_0$=1 and h=0.1

By Euler's algorithm

$y_{n+1} = y_n + h f(x_n,y_n)$ →(1)

$\therefore$ From (1), for n = 0, we have

$y_1 = y_0 + h f(x_0,y_0) = 1+(0.1)f(0,1) = 1+0.1(0-1)$ **= 0.9**

$\therefore$ $\boldsymbol{y_1}$**=0.9**

4. Use Euler's method of find y(0.1),y(0.2) given $y' = (x^3 + xy^2)e^{-x}, y(0) = 1$

Sol: Given $y' = (x^3 + xy^2)e^{-x}, y(0) = 1$

Consider h = 0.1

Here $f(x,y) = (x^3+xy^2)e^{-x}$, $x_0 = 0$, $y_0 = 1$, $x_1=x_0+h=0.1$, $x_2=x_1+h=0.2$

Euler's algorithm is $y_{n+1} = y_n + h\, f(x_n,y_n)$ →(1)

∴ From (1), for n = 0, we have

$y_1 = y_0 + h\, f(x_0,y_0) = y_0+h(x_0^3+x_0y_0^2)e^{-x_0} = 1+(0.1)(0) = 1$

∴ $y(0.1) = 1$

Again $x_2=x_1+h=0.2$

∴ From (1), for n = 1, we have

$y_2 = y_1 + h\, f(x_1,y_1) = y_1 + h(x_1^3+x_1y_1^2)e^{-x_1}$

$= 1 + (0.1)[(0.1)^3 + (0.1)(1)^2] = 1.0091$

∴ $y(0.2) = 1.0091$

5. Given that $\frac{dy}{dx}$ **= xy ,y(0) = 1 determine y(0.1),using Euler's method.**

Sol: The given differentiating equation is $\frac{dy}{dx} = xy$, $y(0) = 1$

a=0, b=0.1

Here $f(x,y) = xy$, $x_0 = 0$ and $y_0 = 1$

Since h is not given much better accuracy is obtained by breaking up the interval (0,0.1) in to five steps.

i.e. $h = \frac{b-a}{5} = \frac{0.1}{5} = 0.02$

Euler's algorithm is $y_{n+1} = y_n + h\, f(x_n,y_n)$ →(1)

∴ From (1) for n = 0, we have

$y_1 = y_0 + h\, f(x_0,y_0) = 1 + (0.02)\, f(0,1) = 1 + (0.02)(0) = 1$

Next we have $x_1 = x_0 + h = 0 + 0.02 = 0.02$

∴ From (1), for n = 1, we have

$y_2 = y_1 + h\, f(x_1,y_1) = 1 + (0.02)\, f(0.02,1) = 1 + (0.02)(0.02) = 1.0004$

Next we have $x_2 = x_1 + h = 0.02 + 0.02 = 0.04$

$\therefore$ From (1), for n = 2,we have

$y_3 = y_2 + h\, f(x_2,y_2) = 1.004 + (0.02)\,(0.04)\,(1.000) = 1.0012$

Next we have $x_3 = x_2 + h = 0.04 + 0.02 = 0.06$

$\therefore$ From (1), for n = 3,we have

$y_4 = y_3 + h\, f(x_3,y_3) = 1.0012 + (0.02)\,(0.06)\,(1.00012) = 1.0024.$

Next we have $x_4 = x_3 + h = 0.06 + 0.02 = 0.08$

$\therefore$ From (1), for n = 4,we have

$y_5 = y_4 + h\, f(x_4,y_4) = 1.0024 + (0.02)\,(0.08)\,(1.00024) = 1.0040.$

Next we have $x_5 = x_4 + h = 0.08 + 0.02 = 0.1$

When $x = x_5$, $y \cong y_5$

$\therefore$ y = 1.0040 when x = 0.1

6. Given that $\frac{dy}{dx} = 3x^2 + y$, y(0) = 4.Find y(0.25) and y(0.5) using Euler's method

Sol: Given $\frac{dy}{dx} = 3x^2 + y$ and y(1) = 2.

Here $f(x,y) = 3x^2 + y$, $x_0 = (1)$, $y_0 = 4$

Consider h = 0.25

Euler's algorithm is $y_{n+1} = y_n + h\, f(x_n,y_n)$ $\rightarrow$(1)

$\therefore$ From (1), for n = 0, we have

$y_1 = y_0 + h\, f(x_0,y_0) = 2 + (0.25)[0 + 4] = 2 + 1 = 3$

Next we have $x_1 = x_0 + h = 0 + 0.25 = 0.25$

When $x = x_1$, $y_1 \simeq y$

$\therefore$ y = 3 when x = 0.25

$\therefore$ From (1), for n = 1, we have

$y_2 = y_1 + h\, f(x_1,y_1) = 3 + (0.25)[3.(0.25)^2 + 3] = 3.7968$

Next we have $x_2 = x_1 + h = 0.25 + 0.25 = 0.5$

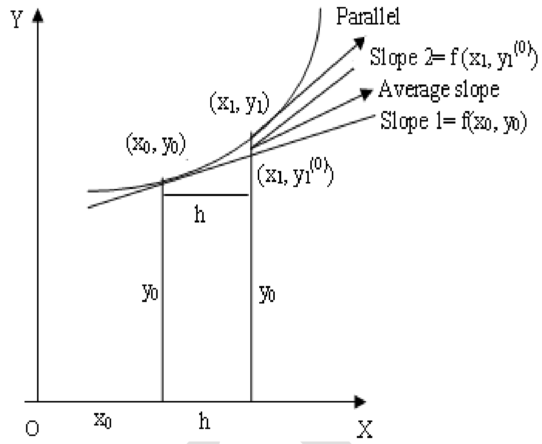
When $x = x_2$, $y \simeq y_2$ $\therefore$ y = 3.7968 when x = 0.5.

MODIFIED EULER'S METHOD

From fig

Avg slope = parallel line slope

$$=\frac{f(x0,y0)+f(x1,y_1^{(0)})}{2}$$



Hence

$y_1^{(0)}$= y$_0$+ hf(x_0, y_0)

$y_1^{(1)}$=y$_0$+$\frac{f(x0,y0)+f(x1,y_1^{(0)})}{2}$

$y_1^{(n+1)}$=y$_0$+ $\frac{f(x_0,y_0)+f(x_1,y_1^{(n)})}{2}$

Continue till any two consecutive iterations nearly same upto three or four decimal places.

To find y_2, y_3.....

The formula is given by

$y_{k+1}^{(i)} = y_k + h/2\left[f(x_k, y_k) + f\left(x_{k+1}, y_{k+1}^{(i-1)}\right)\right], i = 1,2 \ldots. and\ k = 0,1 \ldots\ldots$

Working rule for Modified Euler's method

$$y_{k+1}^{(i)} = y_k + h/2\left[f(x_k, y_k) + f\left(x_{k+1}, y_{k+1}^{(i-1)}\right)\right], i = 1,2 \ldots. and\ k = 0,1 \ldots\ldots$$

ii) When $i=1$ $y^0{}_{k+1}$ can be calculated from Euler's method

iii) K=0, 1……… gives number of iteration. $i=1,2...$

gives number of times, a particular iteration k is repeated

Suppose consider dy/dx=f(x, y) -------- (1) with y(x_0) =y$_0$----------- (2)

To find y (x_1) = y$_1$ at x = x$_1$ = x$_0$+h

Now take k=0 in modified Euler's method

……We get $y_1^{(1)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(i-1)}\right)\right]$……………………… (3)

Taking i=1, 2, 3...k+1 in eqn (3), we get

$y_0^{(0)} = y_0 + h[f(x_0, y_0)]$ (By Euler's method)

$$y_1^{(1)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$

$$y_1^{(2)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

$$y_1^{(k+1)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(k)}\right)\right]$$

If two successive values of $y_1^{(k)}, y_1^{(k+1)}$ are sufficiently close to one another, we will take the common value ……….as $y_1 = y(x_2) = y(x_1 + h)$

Now we have $dy/dx = f(x,y) with\ y = y_1\ at\ x = x_1\ to\ get\ y_2 = y(x_2) = y(x_1 + h)$

Now we have dy/dn =f(x,y) with y=y , at x=x To get y_2=y(x_2)=y($x_1 + h$)

We use the above procedure again

PROBLEMS

1. Using modified Euler's method find the approximate value of x **when** $x = 0.3$ **given that** $dy/dx = x + y\ and\ y(0) = 1$

sol: Given $dy/dx = x + y\ and\ y(0) = 1$

Here $f(x, y) = x + y, x_0 = 0,\ and\ y_0 = 1$

Take h = 0.1 which is sufficiently small

Here $x_0 = 0, x_1 = x_0 + h = 0.1, x_2 = x_1 + h = 0.2, x_3 = x_2 + h = 0.3$

The formula for modified Euler's method is given by

$$y_{k+1}^{(i)} = y_k + h/2\left[f(x_k + y_k) + f\left(x_{k+1}, y_{k+1}^{(i-1)}\right)\right] \to (1)$$

Step1: To find y_1= y(x_1) = y (0.1)

Taking k = 0 in eqn(1)

$$y_1^{(i)} = y_0 + \frac{h}{2}\left[f(x_0 + y_0) + f\left(x_1, y_1^{(i-1)}\right)\right] \quad \to (2)$$

when i = 1 in eqn (2) $y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0 + y_0) + f\left(x_1, y_1^{(0)}\right)\right]$

First apply Euler's method to calculate $y_1^{(0)}$ = y_1

$\therefore \quad y_1^{(0)} = y_0 + h\, f\,(x_0, y_0)$

= 1+(0.1)f(0,1)=1+(0.1)(0+1)

= 1+(0.1) = 1.10

Now $[x_0 = 0, y_0 = 1, x_1 = 0.1, y_1(0) = 1.10]$

$\therefore \quad y_1^{(1)} = y_0 + 0.1/2\left[f\,(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$

= 1+0.1/2[f(0,1) + f(0.1,1.10)

= 1+0.1/2[(0+1)+(0.1+1.10)]= 1.11

When i=2 in eqn (2)

$y_1^{(2)} = y_0 + h/2\left[f\,(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$

= 1+0.1/2[f(0.1)+f(0.1,1.11)]

= 1 + 0.1/2[(0+1)+(0.1+1.11)]= 1.1105

$y_1^{(3)} = y_0 + h/2\left[f\,(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right]$

= 1+0.1/2[f(0,1)+f(0.1 , 1.1105)]

= 1+0.1/2[(0+1)+(0.1+1.1105)] = 1.1105

Since $y_1^{(2)} = y_1^{(3)}$

$\therefore$ y_1 = 1.1105

Step:2 To find $y_2 = y(x_2) = y(0.2)$

Taking k = 1 in eqn (1) , we get

$y_2^{(i)} = y_1 + h/2\left[f\,(x_1, y_1) + f\left(x_2, y_2^{(i-1)}\right)\right] \rightarrow (3)$ where i = 1,2,3,4,…..

For i = 1

$y_2^{(1)} = y_1 + h/2\left[f\,(x_1, y_1) + f\left(x_2, y_2^{(0)}\right)\right]$

$y_2^{(0)}$ is to be calculate from Euler's method

$$y_2^{(0)} = y_1 + h\ f(x_1, y_1)$$

= 1.1105 + (0.1) f(0.1 , 1.1105)

= 1.1105+(0.1)[0.1+1.1105]= 1.2316

$$\therefore\ y_2^{(1)} = 1.1105 + 0.1/2\left[f(0.1, 1.1105) + f(0.2, 1.2316)\right]$$

= 1.1105 +0.1/2[0.1+1.1105+0.2+1.2316]= 1.2426

$$y_2^{(2)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2 y_2^{(1)}\right)\right]$$

= 1.1105 + 0.1/2[f(0.1 , 1.1105) , f(0.2 . 1.2426)]

= 1.1105 + 0.1/2[1.2105 + 1.4426]

= 1.1105 + 0.1(1.3266)= 1.2432

$$y_2^{(3)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2 y_2^{(2)}\right)\right]$$

= 1.1105+0.1/2[f(0.1,1.1105)+f(0.2 , 1.2432)]

= 1.1105+0.1/2[1.2105+1.4432)]

= 1.1105 + 0.1(1.3268) = 1.2432

Since $y_2^{(3)} = y_2^{(3)}$

Hence y_2 = 1.2432

Step:3 To find $y_3 = y(x_3)$ = y y(0.3)

Taking k =2 in eqn (1) we get

$$y_3^{(1)} = y_2 + h/2\left[f(x_2, y_2) + f\left(x_3, y_3^{(i-1)}\right)\right] \rightarrow (4)$$

For i = 1 , $y_3^{(1)} = y_2 + h/2\left[f(x_2, y_2) + f\left(x_3, y_3^{(0)}\right)\right]$

$y_3^{(0)}$ is to be evaluated from Euler's method .

$$y_3^{(0)} = y_2 + h\,f\left(x_2, y_2\right)$$

= 1.2432 +(0.1) f(0.2 , 1.2432)

= 1.2432+(0.1)(1.4432) = 1.3875

$\therefore$ $y_3^{(1)}$ = 1.2432+ $\frac{0.1}{2}$ [f(0.2 , 1.2432)+f(0.3, 1.3875)]

= 1.2432 + 0.1/2[1.4432+1.6875]

= 1.2432+0.1(1.5654) = 1.3997

$$y_3^{(2)} = y_2 + h/2\left[\,f\left(x_2, y_2\right) + f\left(x_3, y_3^{(1)}\right)\right]$$

= 1.2432+0.1/2[1.4432+(0.3+1.3997)]

= 1.2432+ (0.1) (1.575) = 1.4003

$$y_3^{(3)} = y_2 + h/2\left[\,f\left(x_2, y_2\right) + f\left(x_3, y_3^{(2)}\right)\right]$$

= 1.2432+0.1/2[f(0.2 , 1.2432)+f(0.3 , 1.4003)]

= 1.2432 + 0.1(1.5718) = 1.4004

$$y_3^{(4)} = y_2 + h/2\left[\,f\left(x_2, y_2\right) + f\left(x_3, y_3^{(3)}\right)\right]$$

= 1.2432 + 0.1/2[1.4432+1.7004]

= 1.2432+(0.1)(1.5718) = 1.4004

Since $y_3^{(3)} = y_3^{(4)}$

Hence $y_3 = 1.4004$

$\therefore$ The value of y at x = 0.3 is 1.4004

2. Using Modified Euler's method find y(0.2) y(0.4)with h=0.2,given that $\frac{dy}{dx}$=x + siny, y(0)=1

SOL: $f(x,y) = x + siny$ $x_0 = 0; y_0$=1 and h=0.2

Here $x_0 = 0, x_1 = x_0 + h = 0.1, x_2 = x_1 + h = 0.2, x_3 = x_2 + h = 0.3$

$x_1=x_0+h=0.2$; $x_2=x_1+h=0.4$

The formula for modified Euler's method is given by

$$y_{k+1}^{(i)} = y_k + h/2\left[f(x_k + y_k) + f\left(x_{k+1}, y_{k+1}^{(i-1)}\right)\right] \to (1)$$

Step1: To find $y_1= y(x_1) = y\ (0.2)$

Euler's modified method is given by

$y_1^{(1)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$ (k=0 , i=1)

First apply Euler's method to calculate $y_1^{(0)} = y_1$

$\therefore$ $y_1^{(0)} = y_0 + h f(x_0, y_0)$

= 1+(0.2)f(0,1)=1+(0.2)(0+sin1)

= 1.163

Now $\left[x_0 = 0, y_0 = 1, x_1 = 0.2, y_1^{(0)} = 1.163\right]$

$\therefore$ $y_1^{(1)}$ = 1+0.2/2[f(0,1) + f(0.2,1.163)

= 1+0.1/2[1+1.163]

= 1.1916

When i=2 in eqn (2)

$$y_1^{(2)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

= 1+0.2/2[f(0.1)+f(0.2,1.1916)]

=1.2038

$$y_1^{(3)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right]$$

= 1+0.2/2[f(0,1)+f(0.2 , 1.2038)]

=1.2045

Since $y_1^{(2)} = y_1^{(3)}$

$\therefore$ **y_1 = 1.204**

Step:2 To find $y_2 = y(x_2) = y(0.4)$

Taking k = 1 in eqn (1) , we get

$$y_2^{(i)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2, y_2^{(i-1)}\right)\right] \to (3) \text{ where } \quad i = 1,2,3,4,\ldots..$$

For i = 1, $y_2^{(1)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2, y_2^{(0)}\right)\right]$

$y_2^{(0)}$ is to be calculate from Euler's method

$$y_2^{(0)} = y_1 + h\ f(x_1, y_1)$$

= 1.204 + (0.2) f(0.2 , 1.204)

= 1.4313

$y_2^{(1)}$ = 1.204 +0.1[1.1337+1.4313]

= 1.4611

$$y_2^{(2)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2 y_2^{(1)}\right)\right]$$

= 1.204 + 0.1/2[f(0.2 , 1.204) , f(0.4 . 1.416)

= 1.462

$$y_2^{(3)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2 y_2^{(2)}\right)\right]$$

= 1.204+0.1/2[f(0.2,1.204)+f(0.4 , 1.462)] = 1.464

Since $y_2^{(3)} = y_2^{(3)}$

Hence y_2 = 1.46

3. **Using modified Euler's method find the approximate value of** x **when** $x = 0.3$

given that $\frac{dy}{dx} = x - y \ and \ y(0) = 1$

Sol: Given $\frac{dy}{dx} = x - y$ and $y(0) = 1$

Here $f(x,y) = x - y$, $x_0 = 0$ and $y_0 = 1$

Take h = 0.1

Here $x_0 = 0, x_1 = x_0 + h = 0.1, x_2 = x_1 + h = 0.2, x_3 = x_2 + h = 0.3$

Step1: To find $y_1 = y(x_1) = y(0.1)$

First apply Euler's method to calculate $y_1^{(0)} = y_1$

$y_1^{(0)} = y_0 + h f(x_0, y_0)$

= 1+(0.1)(0-1)

= 1-(0.1)

= 0.9

Now $\left[x_0 = 0, y_0 = 1, x_1 = 0.1, y_1^{(0)} = 0.9\right]$

$y_1^{(1)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$

= 1+0.1/2[-1 - 0.8]

= 1-0.09

= 0.91

$y_1^{(2)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$

= 1+0.1/2[-1 + (0.1-0.91)]

= 1+0.1/2[-1.81]

= 1-0.0905

= 0.9095

$y_1^{(3)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right]$

= 1+0.1/2[-1+(0.1-0.9095)]

= 1+0.1/2[-1.8095]

= 1-0.090475

= 0.909525

Since $y_1^{(2)} = y_1^{(3)}$

$\therefore$ $y_1 = 0.9095$

Step:2 To find $y_2 = y(x_2) = y(0.2)$

$y_2^{(0)}$ is to be calculate from Euler's method

$$y_2^{(0)} = y_1 + h\ f(x_1, y_1)$$

= 0.9095+(0.1)(-0.8095)

= 0.82855

$$y_2^{(1)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2, y_2^{(0)}\right)\right]$$

= 0.9095+0.1/2[-0.8095-0.62855]

= 0.9095-0.0719

= 0.8376

$$y_2^{(2)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2 y_2^{(1)}\right)\right]$$

= 0.9095+0.1/2[-0.8095-0.6376]

= 0.9095-0.075355

= 0.837145

$$y_2^{(3)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2 y_2^{(2)}\right)\right]$$

= 0.9095+0.1/2[-10446645]

= 0.9095-0.07233

= 0.83716

Since $y_2^{(3)} = y_2^{(3)}$

Hence $y_2 = 0.8371$

Step:3 To find $y_3 = y(x_3) = y(0.3)$

$y_3^{(0)}$ is to be evaluated from Euler's method

$$y_3^{(0)} = y_2 + h\,f\left(x_2, y_2\right)$$

$= 0.8371+0.1(-0.6371) = 0.7734$

$$y_3^{(1)} = y_2 + h/2\left[f\left(x_2, y_2\right) + f\left(x_3, y_3^{(0)}\right)\right]$$

$= 0.8371+ 0.1/2[-0.6371-0.4734]$

$= 0.8371-0.0555 = 0.7816$

$$y_3^{(2)} = y_2 + h/2\left[f\left(x_2, y_2\right) + f\left(x_3, y_3^{(1)}\right)\right]$$

$= 0.8371+ 0.1/2[-1.1187]$

$= 0.8371-0.056 = 0.7811$

$$y_3^{(3)} = y_2 + h/2\left[f\left(x_2, y_2\right) + f\left(x_3, y_3^{(2)}\right)\right]$$

$= 0.8371+ 0.1/2[-1.1182]$

$= 0.8371-0.05591 = 0.7812$

$$y_3^{(4)} = y_2 + h/2\left[f\left(x_2, y_2\right) + f\left(x_3, y_3^{(3)}\right)\right]$$

$= 0.8371-0.0559 = 0.7812$

Since $y_3^{(3)} = y_3^{(4)}$

Hence $y_3 = 0.7812$

∴ The value of y at x = 0.3 is 0.7812

Runge-Kutta Methods

I.First order R-K Method

EULER'S METHOD is the R-K method of the first order.

II. Second order R-K Method

$y_{i+1} = y_i + 1/2\ (K_1 + K_2)$,

Where $K_1 = h\ (x_i, y_i)$

$K_2 = h\ (x_i + h, y_i + k_1)$

For i= 0,1,2-------

NOTE:EULER'S MODIFIED METHOD IS R-K METHOD OF SECOND ORDER

III. Third order R-K Formula

$y_{i+1} = y_i + 1/6\ (K_1 + 4K_2 + K_3)$,

Where $K_1 = h\ (x_i, y_i)$

$K_2 = h\ (x_i + h/2, y_0 + k_1/2)$

$K_3 = h\ (x_i + h, y_i + 2k_2 - k_1)$ For i= 0,1,2------

IV. Fourth order R-K Formula

$y_{i+1} = y_i + 1/6\ (K_1 + 2K_2 + 2K_3 + K_4)$,

Where $K_1 = h\ (x_i, y_i)$

$K_2 = h\ (x_i + h/2, y_i + k_1/2)$

$K_3 = h\ (x_i + h/2, y_i + k_2/2)$

$K_4 = h\ (x_i + h, y_i + k_3)$

For i= 0,1,2-------

- **Advantages of Runge kutta method Over Taylor series method.**

In RK METHOD no need to find derivatives where as we find derivatives in taylors method. Sometimes it may be complicate to find derivative of some function, sowe go for RK Method at that time.

PROBLEMS:

1. solve $\frac{dy}{dx} = xy$ using R-K method for x=0.2,0.4 given y(0) =1, y`(0)-0 taking h = 0.2

SOL: Given $\frac{dy}{dx} = xy$; $y(0) = 1$.

Here $f(x, y) = xy$, $x_0 = 0$, $y_0=1$ and $h = 0.2$

$\therefore$ $x_1 = x_0+h = 0+0.2 = 0.2.$, $x_2 = x_1+h =0.2+0.2 = 0.4$

By 4th order R-K method, we have

$y_1 = y_0+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$

Where $k_1=h\, f(x_0,y_0)=(0.2)f(0,1)= 0$

$k_2= h\, f(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}) = (0.2)[f(0.1, 1)] = (0.2)(0.1) = 0.02$

$k_3= h\, f((x_0+\frac{h}{2},y_0+\frac{k_2}{2})=(0.2)f(0.1,1.01) = 0.202$

$k_4= h\, f(x_0+h,y_0+k_3)=(0.2)f(0.2,1.202)=0.04808$

Hence $y_1=1+\frac{1}{6}(0+0.04808+2(0.02+0.202)=1.08201$

Step2: To find $y(0.4)= y_2$

Here $x_1 = 0.2$, $y_1=1.08201$ and $h = 0.2$

Again by 4th order R-K method, we have

$\therefore$ $y_2 = y_1+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$

Where $k_1=h\, f(x_1,y_1)=(0.2)[f(0.2,1.08201)] =0.04328$

k_2= hf $(x_1+\frac{h}{2}, y_1+\frac{k_1}{2})$=0.2(f(0.3,1.10364)=0.0662

k_3=hf$(x_1+\frac{h}{2}, y_1+\frac{k_2}{2})$=(0.2)[f(0.3,1.1151)] =0.0669

k_4 =h f$(x_1$+h,$y_1+k_3)$=(0.2)[f(0.4,1.1489)]=0.0919

y_2=1.082+ $\frac{1}{6}$ (0.04328+0.0919+2(0.0662+0.0669)=1.14889

2. Solve the following using R-K fourth method $y' = y - x, y(0) = 2, h = 0.2$ **Find y(0.2).**

SOL: Given $\frac{dy}{dx} = y - x$ **;** y(0) =2

Here f(x, y) = y-x , x_0 = 0, y_0=2 and h = 0.2

∴ $x_1 = x_0+h$ = 0+0.2 = 0.2.

By 4th order R-K method, we have

$y_1 = y_0+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$

Where k_1=h f(x_0,y_0)=(0.2)f(0,2)= 0.2(0.2-0)=0.4

k_2= h f $(x_0+\frac{h}{2}, y_0+\frac{k_1}{2})$

= $(0.2)[f(0.1, 2.2)] = (0.2)(2.2 - 0.1) = 0.42$

k_3= h f($(x_0+\frac{h}{2}, y_0+\frac{k_2}{2})$

=(0.2)f (0.1,2.21)=0.2(2.21-0.1)= 0.422

k_4= h f(x_0+h,y_0+k_3)

=(0.2)f(0.2,2.422)=0.4444

Hence y_1=2+ $\frac{1}{6}$ [0.4+0.4444+2(0.42+0.422)]

∴ $y(0.2)$=2.4214

3. Using Runge-Kutta method of second order, find $y(2.5)$ **from** $\frac{dy}{dx} = \frac{x+y}{x}$ **, y(2)=2 ,**

taking h = 0.25 .

Sol: Given $\frac{dy}{dx} = \frac{x+y}{x}$, y(2) = 2 .

Here $f(x, y) = \frac{x+y}{x}$, $x_0 = 2$, $y_0=2$ and h = 0.25

$\therefore$ $x_1 = x_0+h = 2+0.25 = 2.25$, $x_2 = x_1+h = 2.25+0.25 = 2.5$

By R-K method of second order,

$$y_{i+1} = y_i + 1/2\,(k_1 + k_2), k_1 = hf(x_i, y_i), k_2 = hf(x_i + h, y_i + k_1), i = 0,1 \ldots \longrightarrow (1)$$

Step -1:- To find $y(x_1)$i.e y(2.25) by second order R - K method taking i=0 in eqn(i)

We have $y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

Where $k_1= hf(x_0,y_0)$, $k_2= hf(x_0+h,y_0+k_1)$

$f(x_0,y_0)=f(2,2)=2+2/2=2$

$k_1=hf(x_0,y_0)=0.25(2)=0.5$

$k_2= hf(x_0+h,y_0+k_1)=(0.25)f(2.25,2.5)$

$=(0.25)(2.25+2.5/2.25)=0.528$

$\therefore y_1=y(2.25)=2+1/2(0.5+0.528)=2.514$

Step2:

To find $y(x_2)$ i.e., y(2.5)

i=1 in (1)

$x_1=2.25, y_1=2.514$, and h=0.25

$y_2=y_1+1/2(k_1+k_2)$

where $k_1=h\,f((x_1,y_1)=(0.25)f(2.25,2.514)$

$=(0.25)[2.25+2.514/2.25]=0.5293$

$k_2 = hf(x_1 + h, y_1 + k_1)$

=(0.25)[2.5+2.514+0.5293/2.5]=0.55433

$y_2 = y$ (2.5)=2.514+1/2(0.5293+0.55433)=3.0558

∴ y =3.0558 when x = 2.5

4. Obtain the values of y at x=0.1,0.2 using R-K method of

(i)second order (ii)third order (iii)fourth order for the differential equation y^1+y=0, y(0)=1

Sol: Given dy/dx = -y, y(0)=1

f(x,y) = -y, $x_0 = 0$, $y_0 = 1$

Here f (x,y) = -y, $x_0 = 0$, $y_0 = 1$ take h = 0.1

∴ $x_1 = x_0+h = 0.1$, $x_2 = x_1+h = 0.2$

Second order:

step1: To find $y(x_1)$ i.e y(0.1) or y_1

by second-order R-K method,we have

$y_1 = y_0+1/2(k_1+k_2)$

where $k_1=hf(x_0,y_0)=(0.1)$ f(0,1) = (0.1)(-1)= - 0.1

k_2= hf (x_0+h, y_0+k_1)= (0.1) f (0.1, 1-0.1) = (0.1)(-0.9) = -0.09

y_1=y(0.1)=1+1/2(-0.1-0.09)=1-0.095=0.905

∴ y =0.905 when x=0.1

Step2:

To find y_2 i.e $y(x_2)$ i.e y(0.2)

Here $x_1 = 0.1$, $y_1 = 0.905$ and h=0.1

By second-order R-K method, we have

$y_2 = y(x_2) = y_1 + 1/2(k_1 + k_2)$

Where $k_1 = h\ f(x_1, y_1)$ =(0.1)f(0.1,0.905)=(0.1)(-0.905)=-0.0905

$$k_2 = h\ f(x_1 + h, y_1 + k_1) = (0.1)f(0.2, 0.905 - 0.0905)$$
$$= (0.1)f(0.2, 0.8145) = (0.1)(-0.8145)$$
$$= -0.08145$$

$y_2 =$ y(0.2)=0.905+1/2(-0.0905-0.08145)

= 0.905- 0.085975 = 0819025

(ii) Third order

Step1: To find y_1 i.e $y(x_1)$= y(0.1)

By Third order Runge - Kutta method

$$y_1 = y_0 + 1/6(k_1 + 4k_2 + k_3)$$

where $k_1 = h\ f(x_0, y_0)$ = (0.1) f (0,1) = (0.1) (-1) = -0.1

$$k_2 = h f(x_0 + h/2, y_0 + k_1/2) = (0.1)f(0.1/2, 1 - 0.1/2) = (0.1)f(0.05, 0.95)$$
$$= (0.1)(-0.95) = -0.095$$

and $k_3 = h\ f((x_0+h, y_0+2k_2-k_1)$

=(0.1)[f (0.1,1 + 2(−0.095) + 0.1)] = -0.905

Hence y_1 = 1+1/6(-0.1+4(-0.095)-0.09) = 1+1/6 (-0.57) = 0.905

y_1=0.905 i.e y(0.1)= 0.905

Step2: To find y_2,i.e $y(x_2)$= y(0.2)

Here x_1=0.1,y_1=0.905 and h = 0.1

Again by 3rd order R-K method

$y_2 = y_1 + 1/6(k_1 + 4k_2 + k_3)$

Where k_1=h f(x_1, y_1) = (0.1)f (0.1,0.905)= -0.0905

k_2 = h f (x_1+h/2,y_1+k_1/2)=(0.1)f(0.1+0.05,0.905 - 0.04525)= (0.1) f (0.15, 0.85975)

= (0.1) (-0.85975)= -0.085975

k_3 = h f((x_1+h,y_1+2k_2-k_1)=(0.1)f(0.2,0.905+2(0.085975)+0.0905= -0.082355

y_2 = 0.905+1/6(-0.0905+4(-0.085975)-0.082355)=0.818874

$\therefore$ y = 0.905 when x = 0.1 and y =0.818874 when x =0.2

iii) Fourth order:

step1: x_0=0,y_0=1,h=0.1 To find y_1 i.e y(x_1)=y(0.1)

By 4th order R-K method, we have

y_1 = y_0+1/6(k_1+2k_2+2k_3+k_4)

Where k_1=h f(x_0,y_0)=(0.1)f(0,1)= -0.1

k_2= h f (x_0+h/2, $y_0+\frac{k_1}{2}$) = (0.1)$[f(0.05, 0.95)] = (0.1)(-0.95) = -0.095$

k_3= h f((x_0+h/2,y_0+k_2/2)=(0.1)f (0.1/2,1-0.095/2)=(0.1)(-0.9525)= -0.09525

k_4= h f(x_0+h,y_0+k_3) = (0.1) $[f(0.05, 1 - 0.09525)]$ =(0.1)f(0.05,0.90475) = -0.090475

Hence y_1=1+1/6(-0.1+2(-0.095)+2(0.09525)-0.090475)

=1+1/6(-0.570975)=1-0.0951625 = 0.9048375

Step2: To find $y_2, i.e., y(x_2) = y(0.2), y_1 = 0.9048375, i.e., y(0.1) = 0.9048375$

Here x_1 = 0.1, y_1=0.9048375 and h = 0.1

Again by 4th order R-K method, we have

y_2 = y_1+1/6(k_1+2k_2+2k_3+k_4)

Where k_1=h f(x_1,y_1)=(0.1)$[f(0.1, 0.9048375)]$ =-0.09048375

k_2= hf (x_1+h/2,y_1+k_1/2)=(0.1)$[f(0.1 + 0.1/2, 0.9048375 - 0.09048375/2)]$ =-0.08595956

k_3=hf(x_1+h/2, y_1+k_2/2)=(0.1)$[f(0.15, 0.8618577)]$ = -0.08618577

$k_4 = h\, f(x_1+h, y_1+k_3) = (0.1)[f(0.2, 0.8186517)] = -0.08186517$

Hence $y_2 = 0.9048375 + 1/6(-0.09048375 - 2(0.08595956) - 2(0.08618577) - 0.08186517$

$= 0.9048375 - 0.0861065 = 0.818731$

y = 0.9048375 when x = 0.1 and y = 0.818731 where $x = 0.2$

5. Apply the 4th order R-K method to find an approximate value of y when x=0.2 in steps of 0.1, given that $y^1 = x^2+y^2$, y (1)=1.5

Sol. Given $y^1 = x^2+y^2$, and y(1)=1.5

Here $f(x,y) = x^2+y^2$, $y_0 = 1.5$ and $x_0 = 1$, h=0.1

So that $x_1 = 1.1$ and $x_2 = 1.2$

Step1: To find y_1 i.e., $y(x_1)$

by 4th order R-K method we have

$y_1 = y_0 + 1/6\,(k_1+2k_2+2k_3+k_4)$

$k_1 = hf(x_0,y_0) = (0.1)f(1,1.5) = (0.1)\,[1^2+(1.5)^2] = 0.325$

$k_2 = hf\,(x_0+h/2, y_0+k_1/2) = (0.1)\left[f(1+0.05, 1.5+\frac{0.325}{2})\right] = 0.3866$

$k_3 = hf((x_0+h/2, y_0+k_2/2) = (0.1)f(1.05, 1.5+0.3866/2) = (0.1)[(1.05)^2+(1.6933)^2] = 0.39698$

$k_4 = hf(x_0+h, y_0+k_3) = (0.1)f(1.05, 1.89698) = 0.48085$

Hence

$$y_1 = 1.5 + \frac{1}{6}\left[0.325 + 2(0.3866) + 2(0.39698) + 0.48085\right]$$
$$= 1.8955$$

Step2: To find y_2, i.e., $y(x_2) = y(1.2)$

Here $x_1 = 0.1$, $y_1 = 1.8955$ and h=0.1

by 4th order R-K method we have

$y_2 = y_1+1/6(k_1+2k_2+2k_3+k_4)$

$k_1=hf(x_1,y_1)=(0.1)f(1.10,1.8955)=(0.1)\ [(1.10)^2+(1.8955)^2]=0.48029$

$k_2= hf\ (x_1+h/2,y_1+k_1/2)=(0.1)f(1.1+\frac{0.1}{2},1.8937+\frac{0.4796}{2})\ =0.58834$

$k_3=hf((x_1+h/2,y_1+k_2/2)=(0.1)f(1.15,1.8937+\frac{0.58834}{2})\ =(0.1)[(1.15)^2+(2.189675)^2]=0.611715$

$k_4=hf(x_1+h,y_1+k_3)=(0.1)f(1.2,1.8937+0.610728)=0.77261$

Hence $y_2=1.8955+1/6(0.48029+2(0.58834)+2(0.611715)+0.7726)\ =2.5043$

$\therefore$ y =2.5043 where x=0.2

6. Use R-K method, to approximate y when x=0.2 given that y^1=x+y, y(0)=1

Sol: Here $f(x,y)=x+y, y_0=1, x_0=0$

Since h is not given for better approximation of y

Take h=0.1

$\therefore x_1=0.1,\ x_2=0.2$

Step1 To find y_1 i.e $y(x_1)=y(0.1)$

By R-K method,we have

$y_1=y_0+1/6\ (k_1+2k_2+2k_3+k_4)$

Where $k_1=hf(x_0,y_0)=(0.1)f(0,1)=(0.1)\ (1)=0.1$

$k_2= hf\ (x_0+h/2,y_0+k_1/2)=(0.1)[f(0.05,1.05)]\ =0.11$

$k_3=hf((x_0+h/2,y_0+k_2/2)=(0.1)[f(0.05,1+0.11/2)]\ =(0.1)[(0.05)\ +(1.055)]=0.1105$

$k_4=h\ f\ (x_0+h,y_0+k_3)=(0.1)[f(0.1,1.1105)]\ =(0.1)[0.1+1.1105]=0.12105$

Hence $\therefore\ y_1 = y(0.1) = 1 + \frac{1}{6}(0.1 + 0.22 + 0.2210 + 0.12105)$

y = 1.11034

Step2: To find y_2 i.e $y(x_2) = y(0.2)$

Here $x_1=0.1$, $y_1=1.11034$ and $h=0.1$

Again By R-K method,we have

$y_2=y_1+1/6(k_1+2k_2+2k_3+k_4)$

$k_1=h\ f(x_1,y_1)=(0.1)[f(0.1,1.11034)] =(0.1)\ [1.21034]=0.121034$

$k_2= h\ f\ (x_1+h/2,\ y_1+k_1/2)=(0.1)[f(0.1 + 0.1/2,1.11034 + 0.121034/2)]=0.1320857$

$k_3=h\ f((x_1+h/2,y_1+k_2/2)=(0.1)[f(0.15,1.11034 + 0.1320857/2)]=0.1326382$

$k_4=h\ f(x_1+h,y_1+k_3)=(0.1)[f(0.2,1.11034 + 0.1326382)]=(0.1)(0.2+1.2429783)=0.1442978$

Hence $y_2=1.11034+1/6(0.121034+0.2641714+0.2652764+0.1442978)$

$=1.11034+0.1324631 =1.242803$

$y =1.242803$ when $x=0.2$

7. Compute y(0.1) and y(0.2) by R-K method of 4th order for the D.E. $y^1 = xy+y^2$, y(0)=1

Sol. Given $y^1 = xy+y^2$ **and y(0)=1**

Here $f(x,y)= xy+y^{2,}\ y_0 =1$ and $x_0=0$, $h=0.1$

So that $x_1=0.1$ and $x_2=0.2$

Step1: To find $y_1 = y(x_1) = y(0.1)$

by 4th order R-K method we have

$y_1=y_0+1/6\ (k_1+2k_2+2k_3+k_4)$

$k_1=hf(x_0,y_0)=(0.1)f(0,1)=(0.1)\ [0+1]=0.1$

$k_2= hf\ (x_0+h/2,y_0+k_1/2)=(0.1)[f(0.05, 1.05)] =0.1155$

$k_3=hf((x_0+h/2,y_0+k_2/2)=(0.1)f(0.05,\ 1.05775)=0.11217$

$k_4=hf(x_0+h,y_0+k_3)=(0.1)f(0.1,\ 1.11217)=0.1248$

Hence $y_1 = y(0.1) = y_0 + 1/6[k_1+2k_2+2k_3+k_4]$

$= 1+1/6[0.1+0.0231+0.22434+0.1248]$

$= 1.1133$

Step2: To find $y_2 = y(x_2) = y(0.2)$

Here $x_1=0.1, y_1=1.1133$ and $h=0.1$

by 4th order R-K method we have

$y_2 = y_1+1/6(k_1+2k_2+2k_3+k_4)$

$k_1=hf(x_1,y_1)=(0.1)f(0.1, 1.1133)=0.1351$

$k_2= hf(x_1+h/2,y_1+k_1/2)=(0.1)f(0.15, 1.18085) =0.1571$

$k_3=hf((x_1+h/2,y_1+k_2/2)=(0.1)f(0.15, 1.19185) =0.1599$

$k_4=hf(x_1+h,y_1+k_3)=(0.1)f(0.2, 1.2732) =1.1876$

Hence $y_2 = y(0.2) = y_1 + 1/6[k_1+2k_2+2k_3+k_4]$

$= 1.1133+1/6(0.1351+0.3142+0.3198+0.1876)$

$= 1.2728$

8. Find y(0.1) and y(0.2) by R-K method of 4th order for the D.E. $y^1 = x^2 – y$ and y(0)=1

Sol. Given $y^1 = x^2 – y$ **and y(0)=1**

Here $f(x,y)= x^2 – y$, $y_0 =1$ and $x_0=0$, $h=0.1$

So that $x_1=0.1$ and $x_2=0.2$

Step1: To find $y_1 = y(x_1) = y(0.1)$

by 4th order R-K method we have

$y_1=y_0+1/6 (k_1+2k_2+2k_3+k_4)$

$k_1=hf(x_0,y_0)=(0.1)f(0,1)=(0.1) [0-1]=-0.1$

k_2= hf (x_0+h/2,y_0+k_1/2)=(0.1)[f(0.05, 0.95)] =-0.09475

k_3=hf((x_0+h/2,y_0+k_2/2)=(0.1)f(0.05, 0.952625)=-0.095

k_4=hf(x_0+h,y_0+k_3)=(0.1)f(0.1, 0.905)=-0.0895

Hence $y_1 = y(0.1) = y_0 + 1/6[k_1+2k_2+2k_3+k_4]$

$= 1+1/6[-0.1-0.1895-0.19-0.0895] = 0.9052$

Step2: To find $y_2 = y(x_2) = y(0.2)$

Here x_1=0.1,y_1=0.9052 and h=0.1

by 4^{th} order R-K method we have

$y_2 = y_1+1/6(k_1+2k_2+2k_3+k_4)$

k_1=hf(x_1,y_1)=(0.1)f(0.1, 0.9052)=-0.08952

k_2= hf (x_1+h/2,y_1+k_1/2)=(0.1)f(0.15, 0.86044) =-0.08379

k_3=hf((x_1+h/2,y_1+k_2/2)=(0.1)f(0.15, 0.8633) =-0.0841

k_4=hf(x_1+h,y_1+k_3)=(0.1)f(0.2, 0.8211) =-0.07811

Hence $y_2 = y(0.2) = y_1 + 1/6[k_1+2k_2+2k_3+k_4]$

$= 0.9052+1/6(-0.08952-0.16758-0.1682-0.07811) = 0.8213$

UNIT – III
Fourier series

Fourier series

Suppose that a given function $f(x)$ defined in $[-\pi,\pi]$ (or) $[0,2\pi]$ (or) in any other interval can be expressed as

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

The above series is known as the Fourier series for $f(x)$ and the constants $a_0,a_n,b_n\,(n=1,2,3-----)$ are called Fourier coefficients of $f(x)$

Periodic Function

A function $f(x)$ is said to be periodic with period $T>0$ if for all $x, f(x+T)=f(x)$,and T is the least of such values

Example:-

(1) $\sin x=\sin(x+2\pi)=\sin(x+4\pi)=-----$ the function $\sin x$ is periodic with period 2π. There is no positive value T, $0<T<2\pi$ such that $\sin(x+T)=\sin x\,\forall\, x$

(2) $The\ period\ of\ \tan x\ is\ \pi$

(3) The period of $\sin nx\ is\ \frac{2\pi}{n}\quad i.e\ \sin nx=\sin n\left(\frac{2\pi}{n}+x\right)$

Euler's Formulae

The Fourier series for the function $f(x)$ in the interval $C\le x\le C+2\pi$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

Where $a_0=\frac{1}{\pi}\int_C^{C+2\pi}f(x)dx$

$a_n=\frac{1}{\pi}\int_C^{C+2\pi}f(x)\cos nx.dx$ and

$b_n=\frac{1}{\pi}\int_C^{C+2\pi}f(x)\sin nx.dx$

These values of a_0,a_n,b_n are known as Euler's formulae

Corollary 1: If $f(x)$ is to be expanded as a Fourier series in the interval $0\le x\le 2\pi$, put $C=0$ then the formulae (1) reduces to

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx.dx$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx.dx$$

Corollary 2:- If $f(x)$ is to expanded as a Fourier series in $[-\pi,\pi]$ put $c=-\pi$, the interval becomes $-\pi \le x \le \pi$ and the formulae (1) reduces to

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx$$

Functions Having Points Of Discontinuity

In Euler's formulae for a_0, a_n, b_n it was assumed that $f(x)$ is continuous. Instead a function may have a finite number of discontinuities. Even then such a function is expressible as a Fourier series

Let $f(x)$ be defined by

$$f(x) = \phi(x), c < x < x_0$$
$$= \phi(x), x_0 < x < c + 2\pi$$

Where x_0 is the point of discontinuity in $(c, c+2\pi)$ in such cases also we obtain the Fourier series for $f(x)$ in the usual way. The values of a_0, a_n, b_n are given by

$$a_0 = \frac{1}{\pi}\left[\int_c^{x_0}\phi(x)\,dx + \int_{x_0}^{c+2\pi}\phi(x)\,dx\right]$$
$$a_n = \frac{1}{\pi}\left[\int_c^{x_0}\phi(x)\cos nx.dx + \int_{x_0}^{c+2\pi}\phi(x)\cos nx.dx\right]$$
$$b_n = \frac{1}{\pi}\left[\int_c^{x_0}\phi(x)\sin nx.dx + \int_{x_0}^{c+2\pi}\phi(x)\sin nx.dx\right]$$

Note :-

(i) $$\int_{-\pi}^{\pi}\cos mx\cos nx.\,dx = \begin{cases} 0 \; for\; m \neq n \\ \pi, for\; m = n > 0 \\ 2\pi, for\; m = n > 0 \end{cases}$$

(ii) $$\int_{-\pi}^{\pi}\sin mx\sin nx.\,dx = \begin{cases} 0\; for\;\; m = n = 0 \\ \pi, for\; m \neq n\; > 0 \end{cases}$$

Problems:-

Fourier Series for f(x) in [0, 2π]

1. **Obtain the Fourier series for the function f(x) =** e^x **from x =** [0,2π]

Sol: Let $e^x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$

Then ao $= \frac{1}{\pi}\int_0^{2\Pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\Pi} e^x\,dx = \frac{1}{\pi}(e^x)_0^{2\pi} = \frac{1}{\pi}(e^{2\pi} - 1)$

and an $= \frac{1}{\pi}\int_0^{2\Pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\Pi} e^x \cos nx\,dx$

$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\cos nx + n\sin nx)\right]_0^{2\pi} = \frac{e^{2\pi}-1}{\pi(1+n^2)}$

Finally bn $= \frac{1}{\pi}\int_0^{2\Pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\Pi} e^x \sin nx\,dx$

$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right]_0^{2\pi} = \frac{(-n)(e^{2\pi}-1)}{\pi(1+n^2)}$

Hence $e^x = \frac{e^{2\pi}-1}{\pi} + \sum_{n=1}^{\infty}\frac{e^{2\pi}-1}{\pi(1+n^2)}\cos nx + \sum_{n=1}^{\infty}\frac{(-n)(e^{2\pi}-1)}{\pi(1+n^2)}\sin nx$

$= \frac{e^{2\pi}-1}{2\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\cos nx}{1+n^2} - \sum_{n=1}^{\infty}\frac{n\sin nx}{1+n^2}\right]$

2.Obtain the Fourier series for the function f(x) = x sinx, 0 < x < 2π .

Sol. Given, f(x) = x sinx, 0 < x < 2π

Let x sinx = $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n$ cos nx + $\sum_{n=1}^{\infty} b_n$ sinnx ....(1)

Then $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$

$= \frac{1}{\pi}\int_0^{2\pi} x\, sinx\, dx$

$= \frac{1}{\pi}[x(-cosx) - 1\,(-sinx)]_0^{2\pi}$

$= \frac{1}{\pi}[-xcosx + sinx]_0^{2\pi}$

$= \frac{1}{\pi}[(-2\pi + 0) - (0 + 0)]$

= -2

And $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\, cosnx\, dx$

$= \frac{1}{\pi}\int_0^{2\pi} x\, sinx\, cosnx\, dx$

$= \frac{1}{2\pi}\int_0^{2\pi} x\,(\,2sinx\, cosnx)\, dx$

$= \frac{1}{2\pi}\int_0^{2\pi} x\,\{\sin(n+1)x - \sin(n-1)x\}\,dx$

$= \frac{1}{2\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\}\right]_0^{2\pi} (n\neq 1)$

$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right\}\right] (n \neq 1)$

$= -\frac{1}{n+1} + \frac{1}{n-1}$

$= \frac{2}{n^2-1} \quad (n \neq 1)$

If n = 1, we have

$a_1 = \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx$

$= \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1\left(\frac{-\sin 2x}{4}\right)\right]_0^{2\pi}$

$= \frac{1}{2\pi}(-\pi)$

$= -\frac{1}{2}$

Finally, $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx\,dx$

$= \frac{1}{2\pi}\int_0^{2\pi} x\,(2\sin x\sin nx)\,dx$ …………..(2)

$= \frac{1}{2\pi}\int_0^{2\pi} x\,\{\cos(n-1)x - \cos(n+1)x\}\,dx$

$[2\sin A\sin B = \cos(A-B) - \cos(A+B)]$

$= \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right\} - 1\{-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\}\right]_0^{2\pi} (n\neq 1)$

$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right] (n\neq 1)$

$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right] (n\neq 1)$

Therefore $b_n = 0$ for $n \neq 1$

If n = 1, then

$b_1 = \frac{1}{2\pi}\int_0^{2\pi} x\,2\sin^2 x\,dx$ [Putting n =1 in (2)]

$= \frac{1}{2\pi}\int_0^{2\pi} x\,(1-\cos 2x)\,dx$ [Integration by parts]

$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$

$= \frac{1}{2\pi}\left[2\pi.2\pi - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right]$

$= \pi$

Substituting the values of a_0, a_n and b_n in (1), we get

$$x\sin x = -1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx + \pi\sin x$$

$$= -1 + \pi\sin x - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty}\frac{\cos nx}{n^2-1}$$

This is the required Fourier series.

3. Obtain the Fourier series to represent the function

$f(x) = kx(\pi - x)$ in $0 < x < 2\pi$ Where k is a constant.

Sol. Given $f(x) = kx(\pi - x)$ in $0 < x < 2\pi$ fourier series of the function f(x)

$$f(x) = kx(\pi - x)\ \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{ ---------- } (1)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} kx(\pi - x)\,dx = \frac{k}{\pi}\left[\pi\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{2\pi} = -\frac{2\pi^2 k}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} kx(\pi - x)\cos nx\,dx$$

$$= \frac{k}{\pi}\left[(\pi x - x^2)\frac{\sin nx}{n} - (\pi - 2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{k}{\pi}\left[\left\{0 + \frac{-3\pi}{n^2}\cos 2n\pi + 0\right\} - \left\{0 + \frac{\pi}{n^2} + 0\right\}\right] = \frac{k}{\pi}\left(\frac{-4\pi}{n^2}\right) = -\frac{4k}{n^2}\ (n \neq 0)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} kx(\pi - x)\sin nx\,dx$$

$$= \frac{k}{\pi}\left[(\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{k}{\pi}\left[\left\{\frac{2\pi^2}{n} + 0 - \frac{2}{n^3}\right\} - \left\{0 + 0 - \frac{2}{n^3}\right\}\right] = \frac{2k\pi}{n}$$

put the values of a_0, a_n, b_n in (1) we get

$$f(x) = -\frac{\pi^2 k}{3} - 4k\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx + 2k\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

4. Find the fourier series expansion of the

function $f(x) = \frac{(\pi - x)^2}{4}$ in the interval $0 < x < 2\pi$

Sol: Given $f(x) = \frac{(\pi - x)^2}{4}\ 0 < x < 2\pi$

Fourier series of the function f(x) is given by

$$f(x)=\frac{(\pi-x)^2}{4}=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx---------(1)$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx=\frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{4}dx=\frac{1}{4\pi}\left[\frac{(\pi-x)^3}{3}\right]_0^{2\pi}=\frac{\pi^2}{6}----(2)$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{4}\cos nx\,dx$$

$$=\left[\frac{1}{4\pi}\left[(\pi-x)^2\frac{\sin nx}{n}-\{2(\pi-x)\}\left(-\frac{\cos nx}{n^2}\right)+2\left(-\frac{\sin nx}{n^3}\right)\right]\right]_0^{2\pi}$$

$$=\frac{1}{4\pi}\left[\frac{2\pi}{n^2}+\frac{2\pi}{n^2}\right]=\frac{1}{n^2}-----(3)$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{4}\sin nx\,dx$$

$$=\left[\frac{1}{4\pi}\left[(\pi-x)^2\left(-\frac{\cos nx}{n}\right)-\{2(\pi-x)\}\left(-\frac{\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right]\right]_0^{2\pi}$$

$$=\left[\left(-\frac{\pi^2}{n}+\frac{2}{n^3}\right)-\left(-\frac{\pi^2}{n}+\frac{2}{n^3}\right)\right]=0\,;\,b_n=0-------(4)$$

put the values of a_0, a_n, b_n in (1) we get

$$f(x)=\frac{(\pi-x)^2}{4}=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}=\frac{\pi^2}{12}+\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+\ \ldots.$$

5. Expand $\mathbf{f(x)=\begin{cases}1; & 0<x<\pi\\ 0 & \pi<x<2\pi\end{cases}}$ **as a Fourier Series.**

Sol:- The Fourier series for the function in $(0,2\pi)$ is given

$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx\ldots\ldots(1)$

$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx=\frac{1}{\pi}\left[\int_0^{\pi}1\,dx+\int_{\pi}^{2\pi}0\,dx\right]=\frac{1}{\pi}(x)_0^{\pi}=1$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx.dx=\frac{1}{\pi}\left[\int_0^{\pi}(1)\cos nx.dx+\int_{\pi}^{2\pi}(0)\cos nx.dx\right]$$

$$=\frac{1}{\pi}\left(\frac{\sin nx}{n}\right)_0^{\pi}=\frac{1}{\pi}(0)=0\qquad(\because\sin 0=0,\sin n\pi=0)$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx.\,dx=\frac{1}{\pi}\left[\int_0^{\pi}(1)\sin nx.\,dx+\int_{\pi}^{2\pi}0.\sin nx.\,dx\right]$$

$$=\frac{1}{\pi}\left[\int_0^{\pi}(1)\sin nx.\,dx\right]=\frac{1}{\pi}\left(\frac{-\cos nx}{n}\right)_0^{\pi}$$

$$= -\frac{1}{n\pi}(\cos n\pi - \cos 0) = -\frac{1}{n\pi}[(-1)^n - 1]$$

$$\therefore b_n = \begin{cases} 0 & \text{when n is even} \\ \frac{2}{n\pi} & \text{when n is odd} \end{cases}$$

put the values of a_0, a_n, b_n in (1) we get

$$f(x) = \frac{1}{2} + \frac{2}{n\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{n}\sin nx = \frac{1}{2} + \frac{2}{n\pi}(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ..)$$

6. Obtain Fourier series expansion of $f(x) = (\pi - x)^2$ **in** $0 < x < 2\pi$ **and deduce the value of** $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ----- = \frac{\pi^2}{6}$

Sol:-Given $f(x) = (\pi - x)^2 \; 0 < x < 2\pi$

Fourier series of the function f(x) is given by

$$f(x) = (\pi - x)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad --- (1)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi}(\pi - x)^2 dx$$

$$\frac{1}{\pi}\int_0^{2\pi}[\pi^2 + x^2 - 2\pi x]\,dx = \frac{2\pi^2}{3} \quad ------ (2)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}(\pi - x)^2 \cos nx\,dx$$

$$= \left[\frac{1}{\pi}\left[(\pi - x)^2\frac{\sin nx}{n} - \{2(\pi - x)\}\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right]$$

$$= \frac{4}{n^2} ----(3)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}(\pi - x)^2 \sin nx\,dx$$

$$= \left[\frac{1}{\pi}\left[(\pi - x)^2\left(-\frac{\cos nx}{n}\right) - \{2(\pi - x)\}\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(-\frac{\pi^2}{n} + \frac{2}{n^3}\right) - \left(-\frac{\pi^2}{n} + \frac{2}{n^3}\right)\right] = 0\,;\, b_n = 0 \; -------(4)$$

put the values of a_0, a_n, b_n in (1) we get

$$f(x) = (\pi - x)^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} = \frac{\pi^2}{3} + \frac{4\cos x}{1^2} + \frac{4\cos 2x}{2^2} + \frac{4\cos 3x}{3^2} + ---$$

Deduction:-

Putting $x=0$ in the above equation we get

$$f(0)=(\pi-0)^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}=\frac{\pi^2}{3}+\frac{4\cos 0}{1^2}+\frac{4\cos 0}{2^2}+\frac{4\cos 0}{3^2}+---$$

$$\pi^2=\frac{\pi^2}{3}+\frac{4}{1^2}+\frac{4}{2^2}+\frac{4}{3^2}+---$$

$$\pi^2-\frac{\pi^2}{3}=4\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+---\right]$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+----=\frac{\pi^2}{6}$$

Fourier Series in $[-\pi,\pi]$

1.Express $f(x)=x$ as Fourier series in the interval $-\pi<x<\pi$

Sol. Let the function x be represented by the Fourier series as

$$f(x)=x=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx\to(1)$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=0 \qquad (\because x \text{ is odd function})$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx.\,dx$$

$=0$ ($x\cos nx$ is odd function and $\cos nx$ is even function)

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx.\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(x)\sin nx.\,dx==\frac{1}{\pi}\left[\int_{-\pi}^{\pi}x\ \sin nx.\,dx\right]$$

$$=\frac{1}{\pi}\left[2\int_{-\pi}^{\pi}x\ \sin nx.\,dx\right] \quad [\because x\sin nx \text{ is even function}]$$

$$=\frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right)-1\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}=\frac{2}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n}\right)-(0+0)\right]$$

$(\because \sin n\pi=0, \sin 0=0)$

$$=-\frac{2}{n}\cos n\pi=\frac{-2}{n}(-1)^n=\frac{2}{n}(-1)^{n+1}\ \forall n=1,2,3.....$$

Substituting the values of a_0,a_n,b_n in (1), We get

$$x-\pi=-\pi+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin nx$$

$$=-\pi+2\left[\sin x\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\frac{1}{4}\sin 4x+\ldots\ldots.\right]$$

2**. Obtain the Fourier series for the function f(x) = $|x|$ in $-\pi<x<\pi$ and deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\ldots.=\frac{\pi^2}{8}$**

Sol. Given f(x) = $|x|$ in $-\pi<x<\pi$

Since f(-x) = $|-x|$ = x = $|x|$ = f(x)

Therefore f(x) = $|x|$ is an even function.

Hence the Fourier series will consist of cosine terms only.

Therefore f(x) = $|x| = \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n \cos nx$ …………….(1)

Where $a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx$

$=\frac{2}{\pi}\int_0^{\pi}|x|dx$

$=\frac{2}{\pi}\int_0^{\pi}x\,dx$

$=\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$

$=\frac{2}{\pi}\left[\frac{\pi^2}{2}\right]$

$a_0=\pi$ …………..(2)

And $a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx$

$=\frac{2}{\pi}\int_0^{\pi}|x|\cos nx\,dx$

$=\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx$

$=\frac{2}{\pi}\left[x\frac{\sin nx}{n}-1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$

$=\frac{2}{\pi}\left[\frac{\cos n\pi}{n^2}-\frac{1}{n^2}\right]$

$=\frac{2}{\pi n^2}[(-1)^n-1]$

Therefore $a_n=\begin{cases}0, & \text{if n is even}\\ \frac{4}{n^2}, & \text{if n is odd}\end{cases}$ ………………(3)

Substituting the values of a_0 and a_n from (2) and (3) in (1), we get

$|x|=\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\ldots\ldots\ldots.\right)$ ………(4)

Deduction:

When x=0, $|x|= |0| = 0$

Therefore $x = 0, |x| = |0| = 0$

$\therefore$ Putting x = 0 in (4) ,we have

$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$

$= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

3. Express $f(x) = x - \pi$ as Fourier series in the interval $-\pi < x < \pi$

Sol. Let the function $x - \pi$ be represented by the Fourier series

$$f(x) = x - \pi = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$$

Then, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-\pi)dx \quad = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} xdx = \pi\int_{-\pi}^{\pi} dx\right]$

$$= \frac{1}{\pi}\left[0 - \pi.2\int_0^{\pi} dx\right] \quad (\because \text{x is odd function})$$

$$= \frac{1}{\pi}\left[-2\pi(x)_0^{\pi}\right] \quad = -2(\pi - 0) = -2\pi$$

and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx \quad = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)\cos nx.dx$

$$= \frac{1}{x}\left[\int_{-\pi}^{\pi} x\cos nx.dx - \pi\int_{-\pi}^{\pi}\cos nx.dx\right] \quad = \frac{1}{\pi}\left[0 - 2\pi\int_0^{\pi}\cos nx.dx\right]$$

($x\cos nx$ ▯ is odd function and $\cos nx$ is even function)

$$\therefore a_n = -2\int_0^{\pi}\cos nx.dx = -2\left(\frac{\sin nx}{n}\right)_0^{\pi}$$

$$= \frac{-2}{n}(\sin n\pi - \sin 0) = \frac{-2}{n}(0-0) = 0 \text{ for } n = 1,2,3 \ldots\ldots\ldots.$$

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)\sin nx.dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin nx.dx - \pi\int_{-\pi}^{\pi}\sin nx.dx\right]$$

$$= \frac{1}{\pi}\left[2\int_{-\pi}^{\pi} x\sin nx.dx - \pi(0)\right] \quad [\because x\sin nx \text{ is even function}]$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{2}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n}\right) - (0+0)\right] \quad (\because \sin n\pi = 0)$$

$$= \frac{-2}{\pi}\cos n\pi = \frac{-2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1} \; \forall n = 1,2,3.....$$

Substituting the values of a_0, a_n, b_n in (1),

$$\text{We get } , x - \pi = -\pi + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin nx$$

$$= -\pi + 2\left[\sin x \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \dots \dots\right]$$

4. Find the Fourier series to represent the function e^{-ax} *from* $-\pi \le x \le \pi$.

Deduce from this that $\frac{\pi}{\sinh \pi} = 2\left[\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - - - - -\right]$

Sol. Let the function e^{-ax} be represented by the Fourier series

$$e^{-ax} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$$

Then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}dx = \frac{1}{\pi}\left(\frac{e^{-ax}}{-a}\right)_{-\pi}^{\pi} = \frac{-1}{a\pi}\left(e^{-a\pi} - e^{a\pi}\right) = \frac{e^{a\pi} - e^{-a\pi}}{a\pi}$$

$$\therefore \frac{a_0}{2} = \left[\frac{e^{a\pi} - e^{-a\pi}}{2}\right]\frac{1}{a\pi} = \frac{\sinh a\pi}{a\pi}$$

$$\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx.\,dx = \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}(-a\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$\left[\because \int e^{-ax}\cos bx.\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$\therefore a_n = \frac{1}{\pi}\left\{\frac{e^{-a\pi}}{a^2+n^2}(-a\cos n\pi + 0) - \frac{e^{-a\pi}}{a^2+n^2}(-a\cos n\pi + 0)\right\}$$

$$= \frac{a}{\pi(a^2+n^2)}(e^{a\pi} - e^{-a\pi})\cos n\pi = \frac{2a\cos n\pi \,\sinh a\pi}{\pi(a^2+n^2)}$$

$$= \frac{(-1)^n 2a\,\sinh a\pi}{\pi(a^2+n^2)} (\because \cos n\pi = (-1)^n)$$

$$\text{Finally} b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx.\,dx$$

$$\left[\because \int e^{ax}\sin bx.\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}(-a\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-a\pi}}{a^2+n^2}(0 - n\cos n\pi) - \frac{e^{a\pi}}{a^2+n^2}(0 - n\cos n\pi)\right]$$

$$= \frac{n\cos n\pi(e^{a\pi} - e^{-a\pi})}{\pi(a^2+n^2)} = \frac{(-1)^n 2n\,\sinh a\pi}{\pi(a^2+n^2)}$$

=

Substituting the values of $\frac{a_0}{2}, a_n$ *and* b_n in (1) we get

$$e^{-a\pi} = \frac{\sinh a\pi}{a\pi} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n 2a \sinh a\pi}{\pi(a^2+n^2)}\cos nx + (-1)^n 2n\frac{\sinh a\pi}{\pi(a^2+n^2)}\sin nx\right]$$

$$= \frac{2\sinh a\pi}{a}\left\{\begin{matrix}\left(\frac{1}{2a} - \frac{a\cos x}{1^2+a^2} + \frac{a\cos 2x}{2^2+a^2} - \frac{a\cos 3x}{3^2+a^2} + \cdots\right) \\ \left(\frac{\sin x}{1^2+a^2} - \frac{2\sin 2x}{2^2+a^2} + \frac{3\sin 3x}{3^2+a^2}\ \ldots\right)\end{matrix}\right\} \text{——(2)}$$

Deduction:

Putting $x=0$ and $a=1$ in (2), we get

$$1 = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - - -\right]$$

$$\Rightarrow \frac{\pi}{\sinh\pi} = 2\left(\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - - - -\right)$$

5. Find the Fourier Series of $f(x) = x + x^2, -\pi < x < \pi$

and hence deduce the series

i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + - - - - = \frac{\pi^2}{6}$ ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + - - = \frac{\pi^2}{12}$

Sol: Let $x + x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$

find $a_0 = \frac{1}{\pi}\int_{-\Pi}^{\Pi}(x+x^2)\,dx = \frac{1}{\pi}\left(\frac{x^2}{2} + \frac{x^3}{3}\right)_{\pi}^{-\pi} = \frac{2}{3}\pi^2$

find $a_n = \frac{1}{\pi}\int_{-\Pi}^{\Pi}(x+x^2)\cos nx\,dx$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n} - (1+2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_{-\Pi}^{\Pi}$$

$$= \frac{1}{\pi}\,\square\left[(x+x^2)\frac{\sin nx}{n} - (1+2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_{-\Pi}^{\Pi}$$

$$= \frac{1}{\Pi n^2}[(1+2x)(\cos nx)]_{-\pi}^{\pi}$$

$$= \frac{1}{\Pi n^2}[(1+2\pi)(\cos n\pi) - (1-2\pi)(\cos n\pi)]$$

$$= \frac{1}{\Pi n^2}\,\square\,(4\pi\ \cos n\pi) = \frac{4}{n^2}\ (-1)^n$$

Find $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{-\cos nx}{n} - (1+2x)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi+\pi^2)\frac{-\cos n\Pi}{n} - 0 + 2\left(\frac{\cos n\Pi}{n^3}\right)\right] - \left[(\pi+\pi^2)\frac{-\cos n\pi}{n} - 0 + 2\left(\frac{\cos n\pi}{n^3}\right)\right] = -\frac{2}{n}\ (-1)^n$$

Substituting in (1), the required Fourier series is,

$$x+x^2 = \frac{\pi^2}{3} - 4(\cos x - \cos\frac{2x}{4} + \cos\frac{3x}{9} + ..) + 2(\sin x - \sin\frac{2x}{4} + \sin\frac{3x}{9} + \cdots)$$

6.Obtain the fourier series for $f(x) = x - x^2$ in the interval [-π, π] . Hence show that $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$

Sol: The fourier series of $f(x) = x - x^2$ in [-π, π] is given by

$x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$ ....(1)

Using Euler's formulae , we determine a_n and b_n

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x - x^2\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\, dx - \int_{-\pi}^{\pi} x^2\, dx\right]$

$= \left[0 - 2\int_0^{\pi} x^2\, dx\right]$ ((since x is odd function and x^2 is even function)

$= \frac{-2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} \quad \frac{-2}{3\pi}(\pi^3 - 0) = -\frac{2\pi^2}{3}$

And $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx\, dx$

$= \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} - (1-2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$, integration by parts

$= \frac{-4\cos n\pi}{n^2} = \frac{-4(-1)^n}{n^2}$ $(n \neq 0)$[since $\cos n\pi = (-1)^n$]

Finally $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\sin nx\, dx$

$$= \frac{1}{\pi}\left[(x - x^2)\left(\frac{-\cos nx}{n}\right) - (1-2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi} = \frac{-2\cos n\pi}{n}$$

$$= \frac{-2(-1)^n}{n}$$

Sub the values of a_0 ,a_n and b_n in (1) ,we get

$$x - x^2 = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{-4(-1)^n}{n^2}\cos nx + \frac{-2(-1)^n\sin nx}{n}\right)$$

$$= \frac{-\pi^2}{3} + \sum_{n=1}^{\infty}\left[4\frac{(-1)^{n+1}}{n^2}\cos nx + 2\frac{(-1)^n}{n}\sin nx\right]$$

$$= \frac{-\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots.\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots.\right] \quad ....(2)$$

Deduction:

X = 0 is a point of continuity of f(x). Hence the fourier series of f(x) at

x = 0 converges to f(0)

Putting x = 0 in (2), we get

$0 = \frac{-\pi^2}{3} + 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right)$ i.e., $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

Or $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=\frac{\pi^2}{12}$

7.Find the Fourier series of the periodic function defined

as $\mathbf{f(x)}=\begin{bmatrix}-\pi, & -\pi<x<0\\ x, & 0<x<\pi\end{bmatrix}$ hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+------=\frac{\pi^2}{8}$

Sol. Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx\to(1)$ then

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx\quad=\frac{1}{\pi}\left[\int_{-\pi}^{o}(-\pi)dx+\int_0^{\pi}xdx\right]=\frac{1}{\pi}\left[-\pi(x)_{-\pi}^{0}+\left(\frac{x^2}{2}\right)_0^{\pi}\right]$$

$$=\frac{1}{\pi}\left[-\pi^2+\frac{\pi^2}{2}\right]=\frac{1}{\pi}\left[\frac{-\pi^2}{2}\right]=\frac{-\pi}{2}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx.dx=\frac{1}{\pi}\left[\int_{-\pi}^{o}(-\pi)\cos nx.dx+\int_0^{\pi}x\cos nx.dx\right]$$

$$=\frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0}+\left(x\frac{\sin nx}{n}+\frac{\cos nx}{n^2}\right)_0^{\pi}\right]=\frac{1}{\pi}\left[0+\frac{1}{n^2}\cos n\pi-\frac{1}{\pi n^2}\right]$$

$$=\frac{1}{\pi n^2}(\cos n\pi-1)=\frac{1}{\pi n^2}[(-1)^n_{-1}]$$

$$a_1=\frac{-2}{1^2.\pi},a_2=0,a_3=\frac{-2}{3^2.\pi},a_4=0,a_5=\frac{-2}{5^2.\pi}\ ---$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx.dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx.dx+\int_0^{\pi}x\sin nx.dx\right]$$

$$=\frac{1}{\pi}\left[\pi\left(\frac{\cos nx}{n}\right)_{-\pi}^{0}+\left(-x\frac{\cos nx}{n}+\frac{\sin nx}{n^2}\right)_0^{\pi}\right]$$

$$=\frac{1}{\pi}\left[\frac{\pi}{n}(1-\cos n\pi)-\frac{\pi}{n}\cos n\pi\right]=\frac{1}{n}(1-2\cos n\pi)$$

$b_1=3,b_2=\frac{-1}{2},b_3=1,b_4=\frac{-1}{4}$ and so on

Substituting the values of a_0,a_n *and* b_n in (1), we get

$$f(x)=\frac{-\pi}{4}-\frac{2}{\pi}\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+..\right)+$$
$$\left(3\ \sin x-\frac{\sin 3x}{2}+\frac{3\sin 3x}{3}+\frac{\sin 4x}{4}+.\right)-(2)$$

Deduction: Putting $x=0$ in (2), we obtain $f(0)=\frac{-\pi}{4}-\frac{2}{4}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\ ....\right)..(3)$

Now $f(x)$ is discontinuous at $x=0$

$f(0-0) = -\pi \ and \ f(0+0) = 0$

$f(0) = \frac{1}{2}\left[f(0-0) + f(0+0)\right] = \frac{-\pi}{2}$

Now (3) becomes $\frac{-\pi}{2} = \frac{-\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + ---\right) \to \frac{1}{1^2} + \frac{1}{3^2} + 1/5^2 + \cdots = \frac{\pi^2}{8}$

8. Find the Fourier series of the periodic function defined as $f(x) = \begin{bmatrix} -\pi, & -\pi < x < 0 \\ \pi, & 0 < x < \pi \end{bmatrix}$

Sol. $Let \ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \to (1) \quad then$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \quad = \frac{1}{\pi}\left[\int_{-\pi}^{o}(-\pi)dx + \int_0^{\pi}\pi dx\right]$$

$$= \frac{1}{\pi}\left[-\pi(x)_{-\pi}^{0} + \pi(x)_0^{\pi}\right] = \frac{1}{\pi}\left[-\pi^2 + \pi^2\right] = \frac{1}{\pi}[0] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx \ = \frac{1}{\pi}\left[\int_{-\pi}^{o}(-\pi)\cos nx.dx + \int_0^{\pi}\pi\cos nx.dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \pi\left(\frac{\sin nx}{n}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}(0) \quad (\because \sin 0 = 0, \sin n\pi = 0)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx.dx + \int_0^{\pi}\pi\sin nx.dx\right]$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left(-\pi\frac{\cos nx}{n}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1 - \cos n\pi) - \frac{\pi}{n}(\cos n\pi - \cos 0)\right]$$

$$= \frac{1}{n}(2 - 2\cos n\pi) = \frac{1}{n}(2 - 2(-1)^n) = \begin{cases} 0 \text{ when n is even} \\ \frac{4}{n} \text{ when n is odd} \end{cases}$$

Substituting the values of $a_0, a_n \ and \ b_n$ in (1), we get $f(x) = \sum_{n=1}^{\infty}\frac{4}{n}\sin(nx)$ where n is odd

$$f(x) = 4\left(\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) \pm --\right)$$

Even and Odd Functions:-

A function $f(x)$ is said to be even *if* $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$

Example:-_ $x^2, x^4 + x^2 + 1, e^x + e^{-x}$ are even functions

$x^3, x, \sin x, \cos ecx$ are odd functions

Note1:-

1.Product of two even (or) two odd functions will be an even function

2.Product of an even function and an odd function will be an odd function

Note 2:- $\int_{-a}^{a} f(x)dx = 0$ when $f(x)$ is an odd function

$= 2\int_{0}^{a} f(x)dx$ when $f(x)$ is even function

Fourier series for even and odd functions:-

We know that a function $f(x)$ defined in $(-\pi,\pi)$ can be represented by the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx$

Case (i):- when $f(x)$ is an even function

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$$

Since $\cos nx$ is an even function, $f(x)\cos nx$ is also an even function

Hence $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\ dx$$

Since $\sin nx$ is an odd function, $f(x)\sin nx$ is an odd function

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx = 0$$

$\therefore$ If a function $f(x)$ is even in $(-\pi,\pi)$, its Fourier series expansion contains only cosine terms

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where $a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx.dx, n = 0,1,2,-----$

Case 2:- when $f(x)$ is an odd function in $(-\pi,\pi)$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0$ Since $f(x)$ is odd

Since $\cos nx$ is an even function, $f(x)\cos nx$ is an odd function and hence

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx = 0$$

Since $\sin nx$ is an odd function; $f(x)\sin nx$ is an even function

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

Thus, if a function $f(x)$ defined in $(-\pi, \pi)$ is odd, its Fourier expansion contains only sine terms

$\therefore f(x) = \sum_{n=1}^{\pi} b_n \sin nx$ Where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx.dx$

Problems

1. **1. Expand the function $f(x) = x^2$ as a Fourier series in $[-\pi, \pi]$, hence deduce that**

(i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + - - - - = \frac{\pi^2}{12}$

Sol. Since $f(-x) = (-x)^2 = x^2 = f(x)$

$\therefore f(x)$ is an even function

Hence in its Fourier series expansion, the sine terms are absent

$$\therefore x^2 = \frac{a_0}{2}\sum_{n=1}^{\infty} a_n \cos nx \ldots\ldots (1)$$

Where $a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{2\pi^2}{3}$ …… (2)

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx.dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx.dx = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 + 2\pi\frac{\cos n\pi}{n^2} + 2(0)\right] = \frac{4\cos n\pi}{n^2} = \frac{4}{n^2}(-1)^n \text{------}(3)$$

Substituting the values of a_0 *and* a_n from (2) and (3) in (1) we get

$$x^2=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx=\frac{\pi^2}{3}-4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx$$

$$=\frac{\pi^2}{3}-4\left(\cos x-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\frac{\cos 4x}{4^2}+---\right)\rightarrow(4)$$

Deduction: Putting $x=0$ in (4), we get

$$0=\frac{\pi^2}{3}-4\left(1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+---\right)\Rightarrow 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+----=\frac{\pi^2}{12}$$

2.Find the Fourier series to represent the function $f(x)=|\sin x|,-\pi<x<\pi$

Sol: Since $|\sin x|$ is an even function, $b_n=0$ *for all n*

Let $f(x)=|\sin x|=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx\rightarrow(1)$

Where $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}|\sin x|dx=\frac{2}{\pi}\int_0^{\pi}\sin x\,dx=\frac{2}{\pi}(-\cos x)_0^{\pi}$

$$=\frac{-2}{\pi}(-1-1)=\frac{4}{\pi}$$

and $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx.dx=\frac{2}{\pi}\int_0^{\pi}\sin x.\cos nx\,dx$

$$=\frac{1}{\pi}\int_0^{\pi}[\sin(1+n)x+\sin(1-n)x]\,dx=\frac{1}{\pi}\left[-\frac{\cos(1+n)x}{1+n}-\frac{\cos(1-n)x}{1-n}\right]_0^{\pi}(n\neq 1)$$

$$=-\frac{1}{\pi}\left[\frac{\cos(1+n)\pi}{1+n}+\frac{\cos(1-n)\pi}{1-n}-\frac{1}{1+n}-\frac{1}{1-n}\right]_0^{\pi}(n\neq 1)$$

$$=\frac{-1}{\pi}\left[\frac{(-1)^{n+1}-1}{1+n}+\frac{(-1)^{n+1}-1}{1-n}\right]=\frac{-1}{\pi}\left[(-1)^{n+1}\left\{\frac{1}{1+n}+\frac{1}{1-n}\right\}-\left\{\frac{1}{1+n}+\frac{1}{1-n}\right\}\right]$$

$$=\frac{-1}{\pi}\left[(-1)^{n+1}\frac{2}{1-n^2}-\frac{2}{1-n^2}\right]=\frac{2}{\pi(n^2-1)}[(-1)^{n+1}-1]$$

$$=\frac{-2}{\pi(n^2-1)}[1+(-1)^n]\quad(n\neq 1)$$

$$\therefore a_n=\begin{cases}0 \text{ if n is odd and } n\neq 1\\ \frac{-4}{\pi(n^2-1)} \text{ if n is even}\end{cases}$$

For $n=1, a_1=\frac{2}{\pi}\int_0^{\pi}\sin x.\text{cons}\,x\,dx=\frac{1}{\pi}\int_0^{\pi}\sin 2x\,dx$

$$=\frac{1}{\pi}\left(\frac{-\cos 2x}{2}\right)_0^{\pi}=\frac{-1}{2\pi}(\cos 2\pi-1)=0$$

Substituting the values of a_0, a_1 *and* a_n in (1) We get $|\sin x|=\frac{2}{\pi}+\sum_{n=2,4,--}^{\infty}\frac{-4}{\pi(n^2-1)}\cos nx$

$$=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=2,4---}^{\infty}\frac{\cos nx}{n^2-1}=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1} \quad \text{(Replace n by 2n)}$$

Hence $|\sin x|=\frac{2}{\pi}-\frac{4}{\pi}\left(\frac{\cos 2x}{3}+\frac{\cos 4x}{15}+----\right)$

3. *Show that for* $-\pi<x<\pi, \sin ax=\frac{2\sin a\pi}{\pi}\left[\frac{\sin x}{1^2-a^2}-\frac{2\sin 2x}{2^2-a^2}+\frac{3\sin 3x}{3^2-a^2}-......\right]$

(a **is not an integer**)

Sol: - As sin ax is an Odd function. It's Fourier series expansion will consist of sine terms only

$$\therefore \sin ax=\sum_{n=1}^{\infty}b_n \sin nx---(1)$$

$$\text{where}\quad b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi}\text{sinax}.\sin nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}[\cos(a-n)\text{x}-\cos(\text{a}+\text{n})\text{x}]dx$$

$$[\because 2\sin A\sin B=\cos(\text{A}-\text{B})-\cos(\text{A}+\text{B})]$$

$$b_n=\frac{1}{\pi}\left[\frac{\sin(a-n)x}{a-n}-\frac{\sin(a+n)x}{a+n}\right]_0^{\pi}$$

$$=\frac{1}{\pi}\left[\frac{\sin \text{a}\,\pi\cos \text{n}\,\pi-\cos\pi\sin \text{n}\,\pi}{a-n}-\frac{\text{sina}\,\pi\,\cos\text{n}\,\pi+\cos\text{a}\,\pi\sin\text{n}\,\pi}{a+n}\right]$$

$$b_n=\frac{1}{\pi}\left[\frac{\sin \text{a}\,\pi.\cos \text{n}\,\pi}{a-n}-\frac{\sin a\pi.\cos n\pi}{a+n}\right][\because \sin n\pi=0]$$

$$=\frac{1}{\pi}\sin a\pi\cos n\pi\left(\frac{1}{a-n}-\frac{1}{a+n}\right)$$

$$=\frac{1}{\pi}\sin a\pi\,(-1)^n\left(\frac{a+n-a+n}{a^2-n^2}\right)=\frac{(-1)^n 2n}{\pi(a^2-n^2)}\sin a\pi$$

Substituting these values in (1), we get

$$\sin ax=\frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^n}{(a^2-n^2)}\sin nx=\frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{(n^2-a^2)}\sin nx$$

$$=\frac{2\sin a\pi}{\pi}\left[\frac{\sin x}{1^2-a^2}-\frac{2\sin 2x}{2^2-a^2}+\frac{3\sin 3x}{3^2-a^2}-......\right]$$

4. Find the Fourier series to represent the function $f(x)=\text{Sin x}, -\pi<x<\pi$.

Sol:- since sin x is an odd function $a_0=a_n=0$

Let $f(x) = \sum b_n \sin nx$, where

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin x \sin nx \, dx = \frac{1}{\pi}\int_0^{\pi} [\cos(1-n)x - \cos(1+n)x] \, dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n}\right]_0^{\pi} (n \neq 1) \quad = 0 (n \neq 1)$$

If n=1 $b_1 = \frac{2}{\pi}\int_0^{\pi} \sin^2 x \, dx = \frac{2}{\pi}\int_0^{\pi} \frac{1-\cos 2x}{2} dx = \frac{1}{\pi}\left(x - \frac{\sin 2x}{2}\right)_0^{\pi} = \frac{1}{\pi}(\pi - 0) = 1 \therefore f(x) =$

$b_1 \sin x = \sin x$

5. *Show that for* $-\pi < x < \pi$, $\sin k x = \frac{2\sin k\pi}{\pi}\left[\frac{\sin x}{1^2-k^2} - \frac{2\sin 2x}{2^2-k^2} + \frac{3\sin 3x}{3^2-k^2} - ......\right]$

(k **is not an integer**)

Sol: - As sin kx is an Odd function.

It's Fourier series expansion will consist of sine terms only

$$\therefore \sin k x = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{-------------} (1)$$

Where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} \sin kx . \sin nx \, dx$

$$= \frac{2}{\pi}\int_0^{\pi} [\cos(k-n)x - \cos(k+n)x] dx$$

$[\because 2\sin A \sin B = \cos(A-B) - \cos(A+B)]$

$$b_n = \frac{1}{\pi}\left[\frac{\sin(k-n)x}{k-n} - \frac{\sin(k+n)x}{k+n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin k\pi \cos n\pi - \cos\pi \sin n\pi}{k-n} - \frac{\sin k\pi \cos n\pi + \cos k\pi \sin n\pi}{k+n}\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{\sin k\pi . \cos n\pi}{k-n} - \frac{\sin k\pi . \cos n\pi}{k+n}\right] [\because \sin n\pi = 0]$$

$$= \frac{1}{\pi}\sin k\pi \cos n\pi\left(\frac{1}{k-n} - \frac{1}{k+n}\right) = \frac{1}{\pi}\sin k\pi (-1)^n \left(\frac{k+n-k+n}{k^2-n^2}\right) = \frac{(-1)^n 2n}{\pi(k^2-n^2)}\sin k\pi$$

Substituting these values in (1), we get

$$\sin k x = \frac{2\sin k\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^n}{(k^2-n^2)}\sin nx = \frac{2\sin k\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{(n^2-k^2)}\sin nx$$

$$= \frac{2\sin k\pi}{\pi}\left[\frac{\sin x}{1^2-k^2} - \frac{2\sin 2x}{2^2-k^2} + \frac{3\sin 3x}{3^2-k^2} - ......\right]$$

Half –Range Fourier Series :- **To obtain Fourier series of function f(x) in the interval (o,π)**

1) The sine series

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx \quad where\ b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx.dx$$

2) The cosine series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx$$

$$where\ \ a_0=\frac{2}{\pi}\int_0^{\pi} f(x)dx\ and$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx.dx$$

Note:-

1.Suppose $f(x)=x\ in\ [0,\pi]$.It can have Fourier cosine series expansion as well as Fourier sine series expansion in $[0,\pi]$

2.If $f(x)=x^2\ in\ [0,\pi]$ can have Fourier cosine series expansion as well as Fourier sine series expansion in $[0,\pi]$

Half –Range Fourier Series:-

Problems

1.Find the half range sine series for $f(x)=x(\pi-x), in\ 0<x<\pi$ Deduce that

$$\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+----=\frac{\pi^3}{32}$$

Sol. The Fourier sine series expansion of $f(x)$ in $(0,\pi)$ is

$f(x)=x(\pi-x)=\sum_{n=1}^{\infty} b_n \sin nx$

Where $b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx.dx$; $b_n=\frac{2}{\pi}\int_0^{\pi} x(\pi-x)\sin nx.dx$

$$=\frac{2}{\pi}\int_0^{\pi}\left(\pi x-x^2\right)\sin nx.dx$$

$$=\frac{2}{\pi}\left[\left(\pi x-x^2\right)\left(\frac{-\cos nx}{n}\right)-(\pi-2x)\left(\frac{-\sin nx}{n^2}\right)+(-2)\frac{\cos nx}{n^3}\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\frac{2}{n^3}(1-\cos n\pi)\right]=\frac{4}{n\pi^3}\left(1-(-1)^n\right)$$

$$b_n=\begin{cases}0, \ when \ n \ is \ even\\ \frac{8}{\pi n^3}, \ when \ n \ is \ odd\end{cases}$$

Hence

$$x(\pi-x)=\sum_{n=1,3,5...}\frac{8}{\pi n^3}\sin nx \ (or) \quad x(\pi-x)=\frac{8}{\pi}\left(\sin x+\frac{\sin 3x}{3^3}+\frac{\sin 5x}{5^3}+----\right)\rightarrow(1)$$

Deduction: Putting $x=\frac{\pi}{2}$ in (1), we get

$$\frac{\pi}{2}\left(x-\frac{\pi}{2}\right)=\frac{8}{\pi}\left(sin\frac{\pi}{2}+\frac{1}{3^3}sin\frac{3\pi}{2}+\frac{1}{5^3}sin\frac{5\pi}{2}+---\right)$$

$$\Rightarrow\frac{\pi^2}{4}=\frac{8}{\pi}\left[1+\frac{1}{3^3}sin\left(\pi+\frac{\pi}{2}\right)+\frac{1}{5^3}sin\left(2\pi+\frac{\pi}{2}\right)+\frac{1}{7^3}sin\left(3\pi+\frac{\pi}{2}\right)+---\right]$$

(or) $\frac{\pi^2}{32}=1+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+---$

2.Find the half- range sine series for the function $f(x)=\frac{e^{ax}-e^{-ax}}{e^{a\pi}-e^{-a\pi}} \quad in \ (0,\pi)$

Sol.Let $f(x)=\sum_{n=1}^{\infty}b_n\sin nx$——(1)

Then $b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx.dx=\frac{2}{\pi}\int_0^{\pi}\frac{e^{ax}-e^{-ax}}{e^{a\pi}-e^{-a\pi}}.\sin nx.dx$

$$=\frac{2}{\pi(e^{a\pi}-e^{-a\pi})}\left[\int_0^{\pi}e^{ax}\sin nx.dx-\int_0^{\pi}e^{-ax}\sin nx.dx\right]$$

$$=\frac{2}{\pi(e^{a\pi}-e^{-a\pi})}\left[\left[\frac{e^{ax}}{a^2+n^2}(a\sin nx-n\cos nx)\right]_0^{\pi}-\left[\frac{e^{-ax}}{a^2+n^2}(-a\sin nx-n\cos nx)\right]_0^{\pi}\right]$$

$$=\frac{2}{\pi(e^{a\pi}-e^{-a\pi})}\left[\frac{-e^{a\pi}}{a^2+n^2}n(-1)^n+\frac{n}{a^2+n^2}+\frac{-e^{a\pi}}{a^2+n^2}n(-1)^n-\frac{n}{a^2+n^2}\right]$$

$$=\frac{2n(-1)^n}{\pi(e^{a\pi}-e^{-a\pi})}\left[\frac{e^{-ax}-e^{ax}}{n^2+a^2}\right]=\frac{2n(-1)^{n+1}}{\pi(n^2+a^2)}--——(2)$$

Substituting (2) in (1), we get

$$f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{a^2+n^2}\sin nx=\frac{2}{\pi}\left[\frac{\sin x}{a^2+1^2}-\frac{2\sin 2x}{a^2+2^2}+\frac{3\sin 3x}{a^2+3^2}-----\right]$$

3.Obtain the half-range sine and cosine series for the function

f(x)= $\frac{\pi x}{8}(\pi - x)$ in the range 0≤ $x \le \pi$

Sol**. Half – Range Fourier Sine Series**

The Fourier Sine series of f(x) in (0,π) is

$$f(x) = \frac{\pi x}{8}(\pi - x)$$

$$= \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{……….(1)}$$

Where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{\pi x}{8}(\pi - x) \sin nx \, dx$$

$$= \frac{2}{\pi}\left(\frac{\pi}{8}\right)\int_0^{\pi}(\pi x - x^2) \sin nx \, dx$$

$$= \frac{1}{4}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\frac{2}{n^3}(1 - \cos n\pi)\right]$$

$$= \frac{1}{2}\left[\frac{1-(-1)^n}{n^3}\right]$$

$$b_n = \begin{cases} 0, \text{when n is even} \\ \frac{1}{n^3}, \ \text{when n is odd} \end{cases}$$

Substituting the values of b_n in (1), we get

$$\frac{\pi x}{8}(\pi - x) = \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3} \sin nx$$

$$= \frac{1}{1^3}\sin x + \frac{1}{3^3}\sin 3x + \frac{1}{5^3}\sin 5x + \dots\dots\dots$$

Half – Range Fourier Cosine Series :

The Fourier Sine series of f(x) in (0,π) is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ ……….(2)

Where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{\pi x}{8}(\pi - x)\, dx$$

$$= \frac{1}{4}\int_0^{\pi}(\pi x - x^2)\, dx$$

$$= \frac{1}{4}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right]$$

$$= \frac{\pi^3}{24}$$

And $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{\pi x}{8}\,(\pi - x)\cos nx\,dx$$

$$= \frac{1}{4}\int_0^{\pi}(\pi x - x^2)\cos nx\,dx$$

$$= \frac{1}{4}\left[(\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{1}{4}\left[(\pi x - x^2)\left(\frac{\sin nx}{n}\right) + (\pi - 2x)\left(\frac{\cos nx}{n^2}\right) + (2)\left(\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\left\{0 - \frac{\pi\cos n\pi}{n^2} + 0\right\} - \left\{0 + \frac{\pi}{n^2} + 0\right\}\right]$$

$$= \frac{1}{4}\left[\frac{-\pi\cos n\pi - \pi}{n^2}\right]$$

$$= \frac{-\pi}{4}\left[\frac{1 + \cos n\pi}{n^2}\right]$$

$$= \frac{-\pi}{4}\left[\frac{1 + (-1)^n}{n^2}\right]$$

Therefore $a_n = \begin{cases} 0, \text{When n is odd} \\ \frac{-\pi^2}{2n^2}, \text{when n is even} \end{cases}$

Substituting the values of a_0 and a_n in (2), we get

$$f(x) = \frac{\pi^3}{48} - \frac{\pi}{2}\sum_{n=2,4,6,8\ldots\ldots}^{\infty} \frac{1}{n^2}\cos nx$$

$$= \frac{\pi^3}{48} - \frac{\pi}{2}\left[\frac{1}{2^2}\cos 2x + \frac{1}{4^2}\cos 4x + \frac{1}{6^2}\cos 6x + \cdots \ldots..\right]$$

4 .Find Fourier Cosine and Sine series for the function

$$f(x) = \begin{cases} kx \quad for \quad 0 < x < \frac{\pi}{2} \\ k(\pi - x)\; for \quad \frac{\pi}{2} < x < \pi \end{cases}$$

Sol: Fourier Cosine series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} kx dx + \int_{\frac{\pi}{2}}^{\pi} k(\pi - x)dx\right]$$

$$= \frac{2k}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\frac{\pi}{2}} + \left(\pi x - \frac{x^2}{2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2k}{\pi}\left[\frac{\pi^2}{8} + \left(\pi^2 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right)\right] \quad = \frac{\pi k}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2k}{\pi}\left[\int_0^{\frac{\pi}{2}} x\cos nx\,dx + \int_{\frac{\pi}{2}}^{\pi}(\pi - x)\cos nx\,dx\right]$$

$$= \frac{2k}{\pi}\left[\left\{x\left(\frac{sinnx}{n}\right) - 1\left(\frac{-cosnx}{n^2}\right)\right\}_0^{\pi/2} + \left\{(\pi - x)\left(\frac{sinnx}{n}\right) - (-1)\left(\frac{-cosnx}{n^2}\right)\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2k}{\pi}\left[\frac{\pi}{2}\left(\frac{\sin(\frac{n\pi}{2})}{n}\right) + \left(\frac{\cos(\frac{n\pi}{2})}{n^2}\right) - \frac{1}{n^2} - \frac{(-1)^n}{n^2} - \frac{\pi}{2}\left(\frac{\sin(\frac{n\pi}{2})}{n}\right) + \left(\frac{\cos(\frac{n\pi}{2})}{n^2}\right)\right]$$

$$= \frac{2k}{\pi}\left[\left(2\frac{\cos(\frac{n\pi}{2})}{n^2}\right) - \frac{1}{n^2} - \frac{(-1)^n}{n^2}\right] \quad (n \# 0)$$

$= 0$ for n is odd

$$= \frac{4k}{\pi}\left[\left(\frac{(-1)^{\frac{n}{2}}}{n^2}\right) - \frac{1}{n^2}\right]$$

for n is even (since $cos\frac{n\pi}{2} = 0$ for 'n' is odd and $cos\frac{n\pi}{2} = (-1)^{\frac{n}{2}}$)

∴Fourier Cosine series is given by

$$f(x) = \frac{k\pi}{4} + \frac{4k}{\pi}\sum_{n=2,4....}^{\infty}\left[\left(\frac{(-1)^{\frac{n}{2}}}{n^2}\right) - \frac{1}{n^2}\right] cosnx$$

$$= \frac{k\pi}{4} - \frac{2k}{\pi}\left[\frac{cos\,2x}{1^2} + \frac{cos\,6x}{3^2} + \frac{cos\,10x}{5^2} + \cdots ..\right]$$

Fourier Sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$

$$= \frac{2k}{\pi}\left[\int_0^{\frac{\pi}{2}} x\, sinnx\, dx + \int_{\frac{\pi}{2}}^{\pi}(\pi - x) sinnx\, dx\right]$$

$$= \frac{2k}{\pi}\left[\left\{x\left(\frac{-cosnx}{n}\right) - 1\left(\frac{-sinnx}{n^2}\right)\right\}_0^{\pi/2} + \left\{(\pi - x)\left(\frac{-cosnx}{n}\right) - (-1)\left(\frac{-sinnx}{n^2}\right)\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2k}{\pi}\left[\frac{-\pi}{2}\left(\frac{cos\frac{n\pi}{2}}{n}\right) + \left(\frac{sin\frac{n\pi}{2}}{n^2}\right) + \frac{\pi}{2}\left(\frac{cos\frac{n\pi}{2}}{n}\right) + \left(\frac{sin\frac{n\pi}{2}}{n^2}\right)\right]$$

$= \frac{4k sin\, n\pi/2}{\pi n^2}$ $= 0$ for 'n' is even

$= \frac{4k(-1)^{\frac{n-1}{2}}}{\pi n^2}$ for 'n ' is odd

∴ Fourier Sine series is given by

$$f(x) = \sum_{n=1,3.....}^{\infty} \frac{4k(-1)^{\frac{n-1}{2}}}{\pi n^2} \sin nx = \left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} \ldots ....\right]$$

5. Find Half range Cosine series for the function

$$f(x)=\begin{cases}-k & \text{for } 0<x<\frac{\pi}{2}\\ k & \text{for } \frac{\pi}{2}<x<\pi\end{cases}$$

Sol: Fourier Cosine series is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx$$

Where

$$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)dx \quad =\frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}}(-k)dx+\int_{\frac{\pi}{2}}^{\pi} k dx\right] \quad =\frac{-2k}{\pi}\left[-\frac{\pi}{2}+\pi-\frac{\pi}{2}\right] \quad = 0$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$=\frac{2k}{\pi}\left[-\int_0^{\frac{\pi}{2}} \cos nx\, dx+\int_{\frac{\pi}{2}}^{\pi} \cos nx\, dx\right]$$

$$=\frac{2k}{\pi}\left[\left(\frac{\sin nx}{n}\right)_0^{\frac{\pi}{2}}+\left(\frac{\sin nx}{n}\right)_{\frac{\pi}{2}}^{\pi}\right] \quad =\frac{2k}{\pi}\left[-\left(\frac{\sin\frac{n\pi}{2}}{n}\right)+\left(-\frac{\sin\frac{n\pi}{2}}{n}\right)\right]$$

$$=\frac{-4k}{\pi}\left(\frac{\sin\frac{n\pi}{2}}{n}\right) \quad = 0 \text{ if 'n' is even}$$

$$=\frac{-4k}{\pi n}(-1)^{\frac{n-1}{2}} \text{ if 'n' is odd}$$

[since $\sin\frac{n\pi}{2}=(-1)^{\frac{n-1}{2}}$ if 'n' is odd

$= 0$ if 'n' is even]

∴Fourier Cosine series is given by

$$f(x)=\frac{-4k}{\pi}\sum_{n=1,3,5\ldots\ldots}^{\infty}\frac{1}{n}(-1)^{\frac{n-1}{2}}\cos nx$$

$$=\frac{-4k}{\pi}\left[\frac{\cos x}{1}-\frac{\cos 3x}{3}+\frac{\cos 5x}{5}\ldots\ldots\ldots\ldots..\right]$$

6. Find Fourier Sine series for the function $f(x)=\cos x$ in $0<x<\pi$.

Sol: Fourier Sine series is $f(x)=\sum_{n=1}^{\infty} b_n \sin nx$

Where $b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$

$$=\frac{1}{\pi}\int_0^{\pi} 2\sin nx\cos x\, dx$$

$$=\frac{1}{\pi}\int_0^{\pi}[\sin(n+1)x+\sin(n-1)x]\, dx$$

$$=\frac{1}{\pi}\left[\left(-\frac{\cos(n+1)x}{n+1}\right)+\left(-\frac{\cos(n-1)x}{n-1}\right)\right]_0^{\pi}$$

$$=\frac{1}{\pi}\left[\frac{-(-1)^{n+1}}{n+1}-\frac{-(-1)^{n-1}}{n-1}+\frac{1}{n+1}+\frac{1}{n-1}\right]$$

$= 0$ for 'n' is odd

$= \frac{4n}{\pi(n^2-1)}(n \# 1)$ for 'n' is even.

For n=1

$$b_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\sin x\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(\cos x)\sin x\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\sin 2x\, dx = \frac{1}{\pi}\left(-\frac{\cos 2x}{2}\right)_0^{\pi} = 0$$

∴Fourier Sine series is given by

$$f(x) = \sum_{n=2,4.....}^{\infty}\frac{4n}{\pi(n^2-1)}\sin nx \quad = \frac{8}{\pi}\left[\frac{\sin 2x}{3} + \frac{\sin 4x}{15} + \frac{3\sin 6x}{35}.....\right]$$

7.Find the fourier cosine series of $f(x) = \begin{cases} \cos x, \text{ when } 0 < x < \frac{\pi}{2} \\ 0, \text{ when } \frac{\pi}{2} < x < \pi \end{cases}$

Sol : Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ .....(1)

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}}\cos x\, dx + \int_{\frac{\pi}{2}}^{\pi} 0\, dx\right]$$

$$= \frac{2}{\pi}(\sin x)_0^{\pi/2} = \frac{2}{\pi}(1-0) = \frac{2}{\pi} \quad .....(2)$$

And $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\cos x.\cos nx\, dx + \int_{\pi/2}^{\pi} 0.\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi/2} 2\cos nx\cos x\, dx \quad .....(3)$$

$$= \frac{1}{\pi}\int_0^{\pi/2}[\cos(n+1)x + \cos(n-1)x]dx = \frac{1}{\pi}\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_0^{\pi/2} \quad (n \neq 1)$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right] \quad (n \neq 1)$$

$$= \frac{1}{\pi}\left[\frac{\cos\frac{n\pi}{2}.\sin\frac{\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}.\sin\frac{\pi}{2}}{n-1}\right] \quad (\text{since } \cos\frac{\pi}{2} = 0)$$

$$\therefore a_n = \frac{\cos\frac{n\pi}{2}}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right] \quad (n \neq 1)$$

$$= \frac{-2\cos\frac{n\pi}{2}}{\pi(n^2-1)} \quad (n \neq 1) \quad .....(4)$$

If n=1 ,then $a_1 = \frac{1}{\pi}\int_0^{\pi/2} 2\cos^2 x\, dx$ [from (3)] $= \frac{1}{\pi}\int_0^{\pi/2}(1+\cos 2x)dx = \frac{1}{\pi}\left(x + \frac{\sin 2x}{2}\right)_0^{\pi/2}$

$$= \frac{1}{\pi}\left[\left(\frac{\pi}{2}+0\right) - (0+0)\right] = \frac{1}{2} \quad ....(5)$$

Sub (2), (4) and (5) in (1) , we get

$$f(x) = \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n \cos nx = \frac{1}{\pi} + \frac{1}{2}\cos x - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{\cos\frac{n\pi}{2}}{n^2-1}\cos nx$$

$$= \frac{1}{\pi} + \frac{1}{2}\cos x - \frac{2}{\pi}\left[\frac{\cos\pi}{(2-1)(2+1)}\cos 2x + \frac{\cos\frac{3\pi}{2}}{(3-1)(3+1)}\cos 3x + \frac{\cos 4\pi}{(4-1)(4+1)}\cos 4x + \cdots\right]$$

$$= \frac{1}{\pi} + \frac{1}{2}\cos x - \frac{2}{\pi}\left[-\frac{\cos 2x}{1.3} + \frac{\cos 4x}{3.5} - \frac{\cos 6x}{5.7} + \cdots\right]$$

$$= \frac{1}{\pi} + \frac{1}{2}\cos x + \frac{2}{\pi}\left[\frac{\cos 2x}{1.3} - \frac{\cos 4x}{3.5} + \frac{\cos 6x}{5.7}\right]$$

8.Obtain the fourier cosine series for the function f(x) =xsinx , (0,π)

Sol : Let $f(x) = x\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ ....(1)

Where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} x\sin x\, dx$

$$= \frac{2}{\pi}[x(-\cos x) + (\sin x)]_0^{\pi} = \frac{2}{\pi}[-\pi\cos\pi + \sin\pi] = \frac{2}{\pi}(\pi) = 2$$

And $a_n = \int_0^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\, dx$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x - \sin(n-1)x]\, dx \quad (n \neq 1)$$

$$= \left\{x\left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right] - (1)\left[\frac{-\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right]\right\}_0^{\pi} \quad (n \neq 1)$$

$$= \frac{-1}{(n+1)}\cos(n+1)\pi + \frac{1}{n-1}\cos(n-1)\pi$$

$$= \frac{(-1)^{n-1}}{n-1} - \frac{(-1)^{n+1}}{n+1} = \frac{2(-1)^{n+1}}{n^2-1}, (n \neq 1)$$

$$\therefore a_2 = \frac{-2}{1.3};\ a_3 = \frac{2}{2.4}; a_4 = \frac{-2}{3.5}; a_5 = \frac{2}{4.6}; \ldots$$

Now $a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\, dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\, dx$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - \left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{-\pi}{2}\cos 2\pi\right] = \frac{-1}{2}$$

From (1), we have

$$x\sin x = 1 - \frac{1}{2}\cos x - \frac{2}{1.3}\cos 2x + \frac{2}{2.4}\cos 3x - \frac{2}{3.5}\cos 4x + \cdots \quad .....$$

Deductions: Putting $x = \frac{\pi}{2}$ in (2), we obtain $\frac{\pi}{2} = 1 + \frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} - \frac{2}{7.9} + \cdots$

$$\frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} - \frac{2}{7.9} + \cdots = \frac{\pi}{2} - 1 = \frac{\pi-2}{2}$$

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \cdots. = \frac{\pi-2}{4}$$

Fourier series of $f(x)$ defined in $[c, c+2l]$:-

It can be seen that role played by the functions $1, \cos x, \cos 2x, \cos 3x, \ldots\ldots \sin x, \sin 2x\ldots\ldots\ldots$

In expanding a function $f(x)$ defined in $[c, c+2\pi]$ as a Fourier series, will be played by

$$1, \cos\left(\frac{\pi x}{e}\right), \cos\left(\frac{2\pi x}{e}\right), \cos\left(\frac{3\pi x}{e}\right), \ldots\ldots$$

$$\sin\left(\frac{\pi x}{e}\right), \sin\left(\frac{2\pi x}{e}\right), \sin\left(\frac{3\pi x}{e}\right), \ldots\ldots$$

In expanding a function $f(x)$ defined in $[c, c+2l]$

(i) $\int_c^{c+2l} \sin\left(\frac{m\pi x}{l}\right).\cos\left(\frac{n\pi x}{l}\right) dx = 0$

(ii) $\int_c^{c+2l} \sin\left(\frac{m\pi x}{l}\right).\sin\left(\frac{n\pi x}{l}\right) dx = \begin{cases} 0, & if\ m \neq n \\ l, & \text{if } m = n \neq 0 \\ 0, & \text{if } m = n = 0 \end{cases}$

(iii) $\int_c^{c+2l} \cos\left(\frac{m\pi x}{l}\right).\cos\left(\frac{n\pi x}{l}\right) dx = \begin{cases} 0, & if\ m \neq n \\ l, & \text{if } m = n \neq 0 \\ 2l, & \text{if } m = n = 0 \end{cases}$

[It can be verified directly that, when m, n are integers]

Fourier series of $f(x)$ defined in $[0, 2l]$:-

Let $f(x)$ be defined in $[0, 2l]$ and be periodic with period $2l$. Its Fourier series expansion is defined as $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right]$

Where $a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$, $a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}dx$ and $b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}dx$

Fourier Series Of $f(x)$ Defined In $[-l, l]$

Let $f(x)$ be defined in $[-l, l]$ and be periodic with period $2l$. Its Fourier series expansion is defined as $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$

Where $a_0 = \frac{1}{l}\int_l^{-l} f(x)dx$, $a_n = \frac{1}{l}\int_l^{-l} f(x)\cos\frac{n\pi x}{l}dx$ and $b_n = \frac{1}{l}\int_l^{-l} f(x)\sin\frac{n\pi x}{l}dx$

Fourier series for even and odd functions in $[-l, l]$:-

Let $f(x)$ be defined in $[-l, l]$. If $f(x)$ is even $f(x)\cos\frac{n\pi x}{l}$ is also even

$$\therefore a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

And $f(x)\sin\frac{n\pi x}{l}$ is odd

$$\therefore b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = 0\,\forall n$$

Hence,

If $f(x)$ is defined in $[-l,l]$ and is even its Fourier series expansion is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$where\ a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

If $f(x)$ is defined in $[-l,l]$ and its odd its Fourier series expansion is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad where\ b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Note:- In the above discussion if we put $2l = 2\pi, l = \pi$ we get the discussion regarding the intervals $[0,2\pi]$ and $[-\pi,\pi]$ as special cases

Fourier series of $f(x)$ defined in $[c, c + 2l]$:-

Problems:-

1.Express $f(x) = x^2$ as a Fourier series in $[-l,l]$

Sol: Since $f(-x) = (-x)^2 = x^2 = f(x)$

Therefore $f(x)$ is an even function

Hence the Fourier series of $f(x)$ in $[-l,l]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}\ where\ a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

Hence $a_0 = \frac{2}{l}\int_0^l x^2 dx = \frac{2}{l}\left(\frac{x^3}{3}\right)_0^l = \frac{2l^2}{3}$

and $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$

$$= \frac{2}{l}\left[x^2\left[\frac{\sin\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right] - 2x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$=\frac{2}{l}\left[2x\,\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right]_0^l$$

(Since the first and last terms vanish at both upper and lower limits)

$$\therefore a_n=\frac{2}{l}\left[2l\frac{\cos n\pi}{n^2\pi^2/l^2}\right]=\frac{4l^2\cos n\pi}{n^2\pi^2}=\frac{(-1)^n 4l^2}{n^2\pi^2}$$

Substituting these values in (1), we get

$$x^2=\frac{l^2}{3}+\sum_{n=1}^{\infty}\frac{(-1)^n 4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}=\frac{l^2}{3}-\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{l}$$

$$=\frac{l^2}{3}-\frac{4l^2}{\pi^2}\left[\frac{\cos(\pi x/l)}{1^2}-\frac{\cos(2\pi x/l)}{2^2}+\frac{\cos(3\pi x/l)}{3^2}-----\right]$$

2. Obtain Fourier series for f(x) = x^3 in [-1, 1].

Sol: The given function is x^3 which is odd $a_o=0$, $a_n=0$

$$b_n=\frac{2}{l}\int_0^1 f(x)\sin\frac{n\pi x}{l}\,dx=\frac{2}{l}\int_0^1 x^3\sin n\pi x\,dx$$

$$=2\left[-x^3\frac{\cos n\pi x}{n\pi}+3x^2\sin\frac{n\pi x}{n^2\pi^2}+6x\frac{\cos n\pi x}{n^3\pi^3}-6\sin\frac{n\pi x}{n^4\pi^4}\right]_0^1$$

$$=2\left[\frac{-(-1)^n}{n\pi}+\frac{6(-1)^n}{n^3\pi^3}\right]$$

$$\therefore\ f(x)=2\left[\left(\frac{1}{\pi}-\frac{6}{\pi^3}\right)\sin x+\left(-\frac{1}{2\pi}+\frac{6}{2^3\pi^3}\right)\sin 2\pi x+\left(\frac{1}{3\pi}-\frac{6}{3^3\pi^3}\right)\sin 3\pi x+\left(-\frac{1}{4\pi}+\frac{6}{4^3\pi^3}\right)\sin 4\pi x\right]$$

3.Find the Fourier series of periodicity 3 for f(x) = 2x-x^2 in 0 < x < 3

Sol. Given, f(x) = 2x-x^2 in 0 < x < 3

Here 2l = 3

Therefore l = 3/2

The required Fourier series is of the form

$$f(x)=2x-x^2=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{l}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{l}\right)$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{3}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{3}\right)\quad\ldots\ldots(1)$$

Then $a_0=\frac{1}{l}\int_0^{2l}f(x)dx=\frac{2}{3}\int_0^3(2x-x^2)\,dx$

$$=\frac{2}{3}\left[x^2-\frac{x^3}{3}\right]_0^3$$

$$=\frac{2}{3}(9-9)$$

$$=0$$

$$a_n=\frac{1}{l}\int_0^{2l}f(x)\cos\frac{n\pi x}{l}dx$$

$$=\frac{2}{3}\int_0^3(2x-x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[(2x - x^2)\frac{\sin(\frac{2n\pi x}{3})}{\frac{2n\pi}{3}} - (2 - 2x)\frac{-\cos(\frac{2n\pi x}{3})}{(\frac{2n\pi}{3})^2} + (-2)\frac{-\sin(\frac{2n\pi x}{3})}{(\frac{2n\pi}{3})^3}\right]_0^3$$

$$= \frac{2}{3}\left[\left\{0 - \frac{4\times 9}{4n^2\pi^2} + 0\right\} - \left\{0 + \frac{2\times 9}{4n^2\pi^2} + 0\right\}\right]$$

$$= \frac{-2}{3}\left[\frac{36+28}{4n^2\pi^2}\right]$$

$$= \frac{-9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{3}\int_0^3 (2x - x^2)\sin\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[(2x - x^2)\frac{-\cos(\frac{2n\pi x}{3})}{\frac{2n\pi}{3}} - (2 - 2x)\frac{-\sin(\frac{2n\pi x}{3})}{(\frac{2n\pi}{3})^2} + (-2)\frac{\cos(\frac{2n\pi x}{3})}{(\frac{2n\pi}{3})^3}\right]_0^3$$

$$= \frac{2}{3}\left[\frac{-3}{2n\pi}(-3)\right]$$

$$= \frac{3}{n\pi}$$

Substituting the values of a's and b's in (1), we get

$$2x - x^2 = \frac{-9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left[\frac{2n\pi x}{3}\right]$$

$$= \frac{-9}{\pi^2}\left[\cos\frac{2\pi x}{3} + \frac{1}{2^2}\cos\frac{4\pi x}{3} - \frac{1}{3^2}\cos\frac{6\pi x}{3} + \dots\dots\dots\right]$$

$$+\frac{3}{\pi}\left[\sin\frac{2\pi x}{3} + \frac{1}{2}\sin\frac{4\pi x}{3} + \frac{1}{3}\sin\frac{6\pi x}{3} + \dots\dots\dots\right]$$

4.Find the Fourier series to represent $f(x) = x^2 - 2$, when $-2 \le x \le 2$

Sol. Given, $f(x) = x^2 - 2$

Since function is an even function. Here l= 2

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}\right)$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{2}\right) \qquad \dots\dots\dots\dots(1)$$

Then $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$$= \frac{2}{2}\int_0^2 (x^2 - 2)\,dx$$

$$= \left[\frac{x^3}{3} - 2x\right]_0^2$$

$$= \frac{8}{3} - 4$$

$$= -\frac{4}{3}$$

$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$

$= \frac{2}{2}\int_0^2 (x^2-2)\cos\frac{n\pi x}{2}\,dx$

$= \left[(x^2-2)\left\{\frac{\sin\frac{n\pi x}{2}}{n\pi/2}\right\} - 2x\left\{\frac{-\cos\frac{n\pi x}{2}}{(n\pi)^2/4}\right\} + 2\left\{-\frac{\sin\frac{n\pi x}{2}}{(n\pi)^3/8}\right\}\right]_0^2$

$= \left[\left\{0 + \frac{16}{n^2\pi^2}\cos n\pi + 0\right\} - \{0+0-0\}\right]$

$\therefore a_n = \frac{16}{n^2\pi^2}\cos n\pi$

$= (-1)^n\frac{16}{n^2\pi^2}$

Substituting the values of a_0 and a_n in (1), we get

$f(x) = \frac{-2}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}\cos\frac{n\pi x}{2}$

$x^2 - 2 = \frac{-2}{3} - \frac{16}{\pi^2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}\cos\frac{n\pi x}{2}$

$= \frac{-2}{3} - \frac{16}{\pi^2}\left[\cos\frac{\pi x}{2} - \frac{1}{2^2}\cos\pi x + \frac{1}{3^2}\cos\frac{3\pi x}{2} - \frac{1}{4^2}\cos\frac{4\pi x}{2} + \cdots\right]$

5. Find a Fourier series with period 3 to represent $f(x) = x + x^2$ *in* $(0,3)$

Sol.Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right) \to (1)$

Here $2l = 3$, $l = 3/2$ Hence (1) becomes

$f(x) = x + x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{3} + b_n\sin\frac{2n\pi x}{3}\right) \to (2)$

Where $a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{2}{3}\int_0^3 (x+x^2)\,dx = \frac{2}{3}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_0^3 = 9$

and $a_n = \frac{1}{l}\int_0^2 f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{2}{3}\int_0^3 (x+x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$

Integrating by parts, we obtain

$a_n = \frac{2}{3}\left[\frac{3}{4n^2\pi^2} - \frac{9}{4n^2\pi^2}\right] = \frac{2}{3}\left(\frac{54}{4n^2\pi^2}\right) = \frac{9}{n^2\pi^2}$

$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx \quad = \frac{2}{3}\int_0^3 (x+x^2)\sin\left(\frac{2n\pi x}{3}\right)dx = \frac{-12}{n\pi}$

Substituting the values of a's and b's in (2) we get

$x + x^2 = \frac{9}{2} + \frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{3}\right) - \frac{12}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi x}{3}\right)$

6.If f(x) =|x|, expand f(x)as a fourier series in the interval (–2, 2).

Sol: Here l=2

Since |x| is an even function

∴ The required series is of the form

$|x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$ ....(1)

Where $a_0 = \frac{2}{l}\int_0^l f(x)dx = \int_0^2 |x|\, dx$ (since l = 2)

$= \int_0^2 x\, dx = \left[\frac{x^2}{2}\right]_0^2 = \frac{1}{2}(4-0) = 2$ ....(2)

And $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{2}\, dx = \int_0^2 |x|\cos\frac{n\pi x}{2} dx$ (since l = 2)

$= \int_0^2 x\cos\frac{n\pi x}{2}\, dx$ (Since 0<x<2)

$$= \left[x\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - 1\left[-\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right]\right]_0^2 = \left(0 + \frac{\cos n\pi}{\frac{n^2\pi^2}{4}}\right) - \left(0 + \frac{1}{\frac{n^2\pi^2}{4}}\right)$$

$= \frac{(-1)^n - 1}{\frac{n^2\pi^2}{4}} = \frac{4}{n^2\pi^2}[(-1)^n - 1]$

$$a_n = \begin{cases} 0 \text{ when n is even} \\ \frac{-8}{n^2\pi^2} \text{ when n is odd} \end{cases}$$

$$|x| = 1 - \frac{8}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{2} = 1 - \frac{8}{\pi^2}\left[\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right]$$

7.Find the fourier series of the function f(x),if

$$f(x) = \begin{cases} \frac{1}{2} + x, \ \ when -1 \le x \le 0 \\ \frac{1}{2} - x, \ \ when\ 0 \le x \le 1 \end{cases}$$

Sol: Since $f(-x) = \frac{1}{2} - x$ in $(-1,0) = f(x)$ in $(0,1)$

And $f(-x) = \frac{1}{2} + x$ in $(0,1) = f(x)$in $(-1,0)$

∴ f(x) is an even function

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$ (since l=1)

Then $a_0 = \frac{2}{l}\int_0^1 f(x)\, dx = 2\int_0^1\left(\frac{1}{2} - x\right) dx$

$= 2\left(\frac{x}{2} - \frac{x^2}{2}\right)_0^1 = (x - x^2)_0^1 = (1-1) - (0-0) = 0$

$a_n = \frac{2}{l}\int_0^1 f(x)\cos\frac{n\pi x}{l}\, dx = 2\int_0^1 f(x)\cos n\pi x\, dx$

$= 2\int_0^1\left(\frac{1}{2} - x\right)\cos n\pi x\, dx$

$= 2\left[\left(\frac{1}{2} - x\right)\left(\frac{\sin n\pi x}{n\pi}\right) - (-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$

$$= 2\left[\left(0 - \frac{\cos n\pi}{n^2\pi^2}\right) - \left(0 - \frac{1}{n^2\pi^2}\right)\right]$$

$$= \frac{2}{n^2\pi^2}(1 - \cos n\pi) \quad = \frac{2}{n^2\pi^2}[1 - (-1)^n]$$

$$\therefore \quad a_n = \begin{cases} 0 & \text{if n is even} \\ \frac{4}{n^2\pi^2} & \text{if n is odd} \end{cases}$$

Half- Range Expansion of $f(x) in [0, l]$:-

Some times we will be interested in finding the expansion of $f(x)$ defined in $[0,l]$ in terms of sines only (or) in terms of cosines only. Suppose we want the expansion of $f(x)$ in terms of sine series only. Define $f_1(x) = f(x)$ in $[0,l]$ and $f_1(x) = -f_1(x)\forall n$ with $f_1[2l+x] = f_1(x)$, $f_1(x)$ is an odd function in $[-l,l]$. Hence its Fourier series expansion is given by

$$f_1(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} dx$$

$$where\ b_n = \frac{2}{l}\int_0^1 f_1(x)\,dx$$

The above expansion is valid for $x\ in\ [-l,l]$ in particular for $x\ in\ [0,l]$,

$f_1(x) = f(x)$ and $f_1(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} dx$ where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l} dx$

This expansion is called the half- range sine series expansion of $f(x)$ in $[0,l]$. If we want the half – range expansion of $f(x)$ in $[0,l]$, only in terms of cosines, define $f_1(x) = f(x)$ in $[0,l]$ and $f_1(-x) = f_1(x)$ for all x with

$f_1(x+2l) = f_1(x)$.

Then $f_1(x)$ is even in $[-l,l]$ and hence its Fourier series expansion is given by

$$f_1(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{l}$$

$$where\ a_n = \frac{2}{l}\int_0^l f_1(x)\cos\frac{n\pi x}{l} dx$$

The expansion is valid in $[-l,l]$ and hence in particular on $[0,l]$,

$f_1(x) = f(x)$ hence in $[0,l]$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

Where $a_n = \frac{2}{l}\int_0^1 f(x)\cos\frac{n\pi x}{l}\,dx$

1.The half range sine series expansion of $f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ in $(0, l)$is given by

Where $b_n = \frac{2}{l}\int_0^1 f(x)\sin\frac{n\pi x}{l}dx$

2.The half range cosine series expansion of $f(x)$ *in* $[0,l]$ is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$where\ b_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

Problems:-

1.Find the half- range sine series of $f(x)=1\ in\ [0,l]$

Sol:- The Fourier sine series of $f(x)$ *in* $[0,l]$ is given by $f(x)=1=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

Here $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\int_0^l 1.\sin\frac{n\pi x}{l}dx$

$$= \frac{2}{l}\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right)_0^l = \frac{2}{n\pi}\left[-\cos\frac{n\pi x}{l}\right]_0^l = \frac{2}{n\pi}(-\cos n\pi + 1) = \frac{2}{n\pi}\left[(-1)^{n+1}+1\right]$$

$$\therefore b_n = \begin{cases} 0 & \text{when n is even} \\ \frac{4}{n\pi} & \text{when n is odd} \end{cases}$$

Hence the required Fourier series is $f(x) = \sum_{n=1,3,5---}^{\infty} \frac{4}{n\pi}\sin\frac{n\pi x}{l}$

$$\text{i.e } 1 = \frac{4}{\pi}\left(\sin\frac{n\pi}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + {}^{1}/_{5}\sin\frac{5\pi x}{l} \ldots\ldots.\right)$$

2.Find the half – range cosine series expansion of $f(x)=\sin\left(\frac{\pi x}{l}\right)$ **in** $0<x<l$

Sol. The half-range Fourier Cosine Series is given

$$f(x) = \sin\left(\frac{\pi x}{l}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \ldots\ldots \quad \text{———(1)}$$

Where $a_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}dx \quad = \frac{2}{l}\left[\frac{-\cos\pi x/l}{\pi/l}\right]_0^l = \frac{-2}{\pi}(\cos\pi - 1) = \frac{4}{\pi}$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx = \frac{2}{l}\int_0^l \sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\int_0^l \left[\frac{\sin(n+1)\pi x}{l} - \frac{\sin(n-1)\pi x}{l}\right] dx$$

$$= \frac{1}{l}\left[-\frac{-\frac{\cos(n+1)\pi x}{l}}{(n+1)\pi/l} + \frac{\cos(n-1)\pi x/l}{(n+1)\pi/l}\right]_0^l \quad (n \neq 1)$$

$$= \frac{1}{l}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] \quad (n \neq 1)$$

When n is odd $a_n = \frac{1}{\pi}\left[\frac{-1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] = 0$

When n is even $a_n = \frac{1}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$

$$= \frac{-4}{\pi(n+1)(n-1)} \quad (n \neq 1)$$

If $n = 1, a_1 = \frac{1}{l}\int_0^l 2\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi x}{l}\right) dx = \frac{1}{l}\int_0^l \sin\left(\frac{2\pi x}{l}\right) dx$

$$= \frac{1}{l}\cdot\frac{1}{2\pi}\left[-\cos\left(\frac{2\pi x}{l}\right)\right]_0^l = \frac{-1}{2\pi}(\cos 2\pi - \cos 0) = \left({}^{-1}/_{2\pi}\right)(1-1) = 0$$

from equation(1) we have

$$\therefore \sin\left(\frac{\pi x}{l}\right) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos(2\pi x/l)}{1.3} + \frac{\cos(4\pi x/l)}{3.5} + ---\right]$$

3. Obtain the half range cosine series for $f(x) = x - x^2$, $0 \leq x \leq 1$.

Sol:The half range cosine series for f(x) in $0 \leq x \leq 1$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

Where $a_0 = \frac{2}{1}\int_0^1 f(x)\, dx = 2\int_0^1 (x - x^2)dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

$$a_n = 2\int_0^1 (x - x^2)\cos n\pi x\, dx$$

$$= 2\left[(x - x^2)\frac{\cos n\pi x}{n\pi} + (1-2x)\frac{\cos n\pi x}{n\pi^2}\right]_0^1$$

$$= 2\left[(-1)\frac{\cos n\pi x}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right] = 2\left[\frac{(-1)^{n+1} - 1}{n^2\pi^2}\right]$$

∴ The cosine series of f(x) is given by,

$$f(x) = \frac{1}{6} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1} - 1}{n^2}\right\}\cos n\pi x = \frac{1}{6} - \frac{4}{\pi^2}\left\{\frac{\cos 2\pi x}{2^2} + \frac{\cos 4\pi x}{4^2} + ----\right\}$$

4. Obtain the half range sine series for e^x in $0 < x < 1$.

Sol: The sine series is $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

Where $bn = \frac{2}{l}\int_0^1 f(x)\sin\frac{n\pi x}{l}\, dx$

$$= 2\int_0^1 e^x \sin n\pi x\, dx$$

$$= \left[\frac{2e^x}{(1+n^2\pi^2)}[\sin n\pi x - n\pi x \cos n\pi x]\right]_0^1$$

$$= \frac{2}{(1+n^2\pi^2)}[-n\pi e \cos n\pi \ + n\pi] = \frac{2}{(1+n^2\pi^2)}[1 - e(-1)^n]$$

$$\therefore e^x = 2\pi \left[\frac{(1+e)}{1+\pi^2} \sin \pi x + \frac{2(1-e)}{1+4\pi} \sin 2\pi x + \frac{3(1+e)}{1+9\pi} \sin 3\pi x + - - - \right]$$

UNIT-IV
PARTIAL DIFFERENTIAL EQUATIONS

Definition:

A Differential equation involves a dependent variable and its derivatives with respective to two or more independent variables is called Partial Differential Equation.

$$\text{Ex: } x\frac{\partial z}{\partial y} + 4y\frac{\partial z}{\partial x} = 2z + 3xy$$

LINEAR & NON LINEAR PARTIAL DIFFERENTIAL EQUATION

If the partial derivatives of the dependent variable occur in first degree only and separately, Such a Partial Differential Equation is called
as linear Partial Differential Equation Otherwise it is called as non –
linear Partial Differential Equation.

HOMOGENEOUS & NON HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATION

A Partial Differential Equation is said to be Homogeneous if each term of the equation

contains either the dependent variable or one of its derivatives Otherwise it is called as a Non - Homogeneous Partial Differential Equation.

FORMATION OF PARTIAL DIFFERENTIAL EQUATION

Partial Differential equation can be formed by two methods , there are

- By the elimination of arbitrary constants
- By the elimination of arbitrary functions

BY ELIMINATION OF ARBITRARY CONSTANTS

Let the given function be $f(x, y, z, a, b) = 0$ ……….(1)

Where a and b are arbitrary constants.

eliminate a and b from equation (1) , by differentiating (1) partially w.r.t. 'x' and 'y'

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}.\frac{\partial z}{\partial x}=0 \Rightarrow \frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}.p=0 \cdots\cdots (2)$$

And
$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}.\frac{\partial z}{\partial y}=0 \Rightarrow \frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}.q=0 \cdots\cdots (3)$$

Now eliminate the constants a and b from (1), (2) and (3). We get a partial differential equation of the first order of the form. $\phi(x, y, z, p, q) = 0$.

Note : 1. If the number of arbitrary constants is equal to the number of variables, a partial differential equation of first order can be obtained.

2.If the number of arbitrary constants is greater than the number of variables, a partial differential equation of order higher than one can be obtained.

PROBLEM

1. **Form the partial differential equation by eliminating the arbitrary constants a and b from $z = ax + by + ab$**

Sol. Given equation is $z = ax + by + ab$ ……….(1)

Differentiating (1) partially w.r.t. x and y, we get

$$\frac{\partial z}{\partial x} = a \Rightarrow p = a \text{ ……… (2) and}$$

$$\frac{\partial z}{\partial y} = b \Rightarrow q = b \text{ …………(3)}$$

Substituting the values of a and b in equation (1), we get

$$z = px + qy + pq$$

Hence the required partial differential equation is

$$\boxed{z = px + qy + pq}$$

2. **Form the partial differential equation by eliminating the arbitrary constants a and b from**

(a) $z = ax + by + a^2 + b^2$

(b) $z = ax + by + \frac{a}{b} - b$

Sol.

(a) Given equation is $z = ax + by + a^2 + b^2$ .... (1)

Differentiating (1) partially w.r.t. x and y, we get

$$\frac{\partial z}{\partial x} = a \Rightarrow p = a \text{ ………… (2) and}$$

$$\frac{\partial z}{\partial y} = b \Rightarrow q = b \text{ …………..(3)}$$

Substituting the values of a and b in equation (1), we get

$$z = px + qy + p^2 + q^2$$

Hence the required partial differential equation is

$$\boxed{z = px + qy + p^2 + q^2}$$

(b) Given equation is $z = ax + by + \frac{a}{b} - b$ ....(1)

Differentiating (1) partially w.r.t. x and y, we get

$$\frac{\partial z}{\partial x} = a \Rightarrow p = a \text{ ....(2) and}$$

$$\frac{\partial z}{\partial y} = b \Rightarrow q = b \text{ .... (3)}$$

Substituting the values of a and b in equation (1), we get

$$z = px + qy + \frac{p}{q} - q$$

Hence the required partial differential equation is

$$\boxed{z = px + qy + \frac{p}{q} - q}$$

3. Form the partial differential equation by eliminating the arbitrary constants from $(x-a)^2 + (y-b)^2 + z^2 = r^2$

(OR)

Find the differential equation of all spheres of fixed radius having their centre on the xy **plane.**

Sol. The equation of sphere of radius r having their centers on xy-plane is

$$(x-a)^2 + (y-b)^2 + z^2 = r^2 \text{ ....(1)}$$

Differentiating (1) partially w.r.t. x and y , we get.

$$2(x-a) + 2z.\frac{\partial z}{\partial x} = 0 \Rightarrow (x-a) + zp = 0 \, or \, x - a = -zp \rightarrow (2)$$

$$2(y-b) + 2z.\frac{\partial z}{2y} = 0 \, or \, (y-b) + zq = 0 \, or \, y - b = -zq \rightarrow (3)$$

Substituting the values of $(x-a)$ *and* $(y-b)$ from (2) and (3) in (1), we get

$$(-zp)^2 + (-zq)^2 + z^2 = r^2$$

$$or \, z^2 (p^2 + q^2 + 1) = r^2$$

Hence the required partial differential equation is

$$\boxed{(-zp)^2 + (-zq)^2 + z^2 = r^2}$$

4. **Form the partial differential equation by eliminating the arbitrary constants a and b from $z = (x+a)(y+b)$**

Sol.The given equation $is\ z = (x+a)(y+b) \ldots\ldots\ldots\ldots.(1)$

Differentiating (1) w.r.t., x

$$p = \frac{\partial z}{\partial x} = 1\,(y + b) \ldots\ldots\ldots.\ldots\ldots (2)$$

Differentiating (1) w.r.t., x

$$q = \frac{\partial z}{\partial y} = 1\,(x + a) \ldots\ldots\ldots\ldots\ldots (3)$$

$$\text{from } (2),(3)\ P = (y + b)\ ,\ q = (x + a)$$

Substituting in (1) we get $z = pq$

Hence the required partial differential equation is

$$\boxed{z = pq}$$

5. **Form the partial differential by eliminating the arbitrary constants from**

$\log(az-1) = x + ay + b$

Sol. Given equation is $\log(az-1) = x + ay + b$ …………(1)

Differentiating (1) partially $w.r.t.\,x\ and\ y$, we get

$$\frac{1}{(az-1)}.a.\frac{\partial z}{\partial x} = 1\,or\,\frac{1}{(az-1)}ap = 1\,or\,ap = az-1 \ldots.(2)$$

and $$\frac{1}{(az-1)}a.\frac{\partial z}{\partial y} = a \Rightarrow aq = (az-1)a \ldots\ldots\ldots\ldots\ldots\ldots.(3)$$

$$(3)\div(2),\,gives\,\frac{q}{p} = a \Rightarrow ap = q \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4)$$

Substituting (4) in equation (2), we get

$$q = \frac{q}{p}z - 1\,or\,pq = qz - p\,or\,p(q+1) = q^2$$

Hence the required partial differential equation is

$$\boxed{q = \frac{q}{p}z - 1\,or\,pq = qz - p\,or\,p(q+1) = q^2}$$

6. **Form the differential equation by eliminating a and b from**

$2z = (x+a)^{\frac{1}{2}} + (y-a)^{\frac{1}{2}} + b$

Sol. Given equation is $2z = (x+a)^{\frac{1}{2}} + (y-a)^{\frac{1}{2}} + b$ …………..(1)

Differentiating (1) partially w.r.t. x and y, we have,

$$2\frac{\partial z}{\partial x} = 2p = \frac{1}{2\sqrt{x+a}} \Rightarrow \frac{1}{\sqrt{x+a}} = 4p$$

$$\sqrt{x+a} = \frac{1}{4p}, \quad x+a = \frac{1}{16p^2} \rightarrow \ldots\ldots\ldots(2)$$

And $$2\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{y-a}} \; or\; 2q = \frac{1}{2\sqrt{y-a}} \; or \sqrt{y-a} = \frac{1}{4q}$$

$$\therefore y-q = \frac{1}{16q^2} \rightarrow \ldots\ldots\ldots\ldots (3)$$

Adding (2) and (3), we get

$$x+y = \frac{1}{16}\left(\frac{1}{p^2}+\frac{1}{q^2}\right)$$

$$or\, 16(x+y)p^2q^2 = p^2+q^2$$

Hence the required partial differential equation is

$$\boxed{or\, 16(x+y)p^2q^2 = p^2+q^2}$$

7. Form the partial differential equation by eliminating the arbitrary constants a and b from $z = ax^3 + by^3$

Sol. Given equation is $z = ax^3 + by^3 \rightarrow$ (1)

Differentiating (1) partially w.r.t. x and y, we get

$$\frac{\partial z}{\partial x} = 3ax^2 \; or\; p = 3ax^2 \Rightarrow a = \frac{p}{3x^2} \rightarrow \quad (2)$$

And $$\frac{\partial z}{\partial y} = 3by^2 \; or\; q = 3by^2 \Rightarrow b = \frac{q}{3y^2} \rightarrow \quad (3)$$

Substituting the values of ‘a’ and ‘b’ from (2) and (3) in equation (1), we get

$$z = \frac{p}{3}x + \frac{q}{3}y$$

Hence the required partial differential equation is

$$\boxed{3z = px+qy}$$

8. **Form the partial differential equation by eliminating the arbitrary constants a and b from** $\boldsymbol{z = (x^2+a)(y^2+b)}$

Sol. The given equation is $z = (\boldsymbol{x^2+a})(\boldsymbol{y^2+b}) \ldots\ldots\ldots\ldots (1)$

Differentiating (1) w.r.t., x

$$p = \frac{\partial z}{\partial x} = \boldsymbol{2x(y^2 + b)} \dots\dots\dots\dots\dots\dots\dots (2)$$

$$\therefore (\boldsymbol{y^2 + b}) = \frac{p}{2x}$$

Differentiating (1) w.r.t., y,we get

$$q = \frac{\partial z}{\partial y} = 2y\,(\boldsymbol{x^2 + a}) \dots\dots\dots\dots\dots\dots\dots. (3)$$

$$\therefore (\boldsymbol{x^2 + a}) = \frac{q}{2y}$$

Substituting in (1) we get $z = \frac{pq}{4xy}$

Hence the required partial differential equation is

$$\boxed{pq - 4xyz = 0}$$

9. Form the partial differential equation by eliminating the arbitrary constants from $(x - a)^2 + (y - b)^2 = z^2 \cot^2 \alpha$

Sol. Given equation is $(x - a)^2 + (y - b)^2 = z^2 \cot^2 \alpha$........(1)

Differentiating (1) partially w.r.t. x

$$(x - a) = z\, p \cot^2 \alpha \;\dots\dots\dots\dots\dots\dots\dots (2)$$

Differentiating (1) partially w.r.t. y

$$(y - b) = z\, q \cot^2 \alpha \;\dots\dots\dots\dots\dots\dots\dots (3)$$

Substituting (2),(3) in equation (1),we get

$(z\, p \cot^2 \alpha)^2 + (z\, q \cot^2 \alpha)^2 = z^2 \cot^2 \alpha$

∴The required Partial differential equation is

$$\boxed{p^2 + q^2 = \tan^2 \alpha}$$

Formation of the Partial Differential Equation By The Elimination Of Arbitrary Functions

Derive a partial differential equation by the elimination of the arbitrary function ϕ from $\phi(u, v) = 0$ where u, v are functions of x, y and z..

$$\phi(u, v) = 0 \;\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1)$$

Differentiating (1) partially w.r.t. x *and* y, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}.\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}.\frac{\partial z}{\partial x}\right) = 0$$

i.e., $$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0\ldots\ldots(2)$$

and $$\frac{\partial\phi}{\partial 4}\left(\frac{\partial 4}{\partial y}+q\frac{\partial 4}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0\ldots\ldots..(3)$$

Eliminating $\frac{\partial\phi}{\partial u}$ and $\frac{\partial\phi}{\partial v}$ from equations (2) and (3) ,we get

$$\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}p\right)\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)$$

i.e. $$\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}+\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$\frac{\partial(u,v)}{\partial(y,z)}p+\frac{\partial(u,v)}{\partial(z,x)}q=\frac{\partial(u,v)}{\partial(x,y)}$$

Above equation is generally written as $Pp+Qq=R$ where

$$P=\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y},\ Q=\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\ and\ \ R=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

PROBLEMS

1.Form the partial differential equation by eliminating a,b,c from $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Sol. Given equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ……. (1)

Differentiating (1) partially w.r.t. x and y.

$$\frac{2x}{a^2}+\frac{2z}{c^2}.p=0\ or\ \frac{x}{a^2}+\frac{z}{c^2}.p=0\rightarrow\ (2)$$

And $$\frac{2y}{b^2}+\frac{2z}{c^2}.q=0\ or\ \frac{y}{b^2}+\frac{z}{c^2}.q=0\rightarrow\quad(3)$$

Since it is not possible to eliminate a, b,c from equations (1), (2) and (3).

Differentiating (2), partially w.r.t. ‘x’, we get

$$\frac{1}{a^2}+\frac{1}{c^2}\left(z.\frac{\partial p}{\partial x}+p.\frac{\partial z}{\partial x}\right)=0\ or\ \frac{1}{a^2}+\frac{1}{c^2}.z.\frac{\partial^2 z}{\partial x^2}+\frac{1}{c^2}.p$$

$$\therefore\frac{1}{a^2}+\frac{1}{c^2}.zr+\frac{p^2}{c^2}=0\rightarrow\qquad(4)$$

Multiplying (4) by ‘x’ and then subtracting (2) from it, we get

$$\frac{xz}{c^2}.r+\frac{xp^2}{c^2}-\frac{z}{c^2}.p=0\ or\ \frac{1}{c^2}\left(xzr+xp^2-zp\right)=0$$

∴The required Partial differential equation is

$$\therefore pz = xp^2 + xzr$$

2. Form a partial differential equation by eliminating the arbitrary the function

$$\boldsymbol{\varphi(x^2 + y^2, z - xy) = 0}$$

Sol. Given *equation is* $\varphi(x^2 + y^2, z - xy) = 0$

This can be written as $z - xy = f(x^2 + y^2)$-------------(1)

Now we have to eliminate f from (1)

Differentiating (1) partially w.r.t., x

$$\frac{\partial z}{\partial x} - y = f'(x^2 + y^2)(2x)$$

$$p - y = f'(x^2 + y^2)(2x) \text{--------(2)}$$

Differentiating (2) partially w.r.t., y

$$q - x = f'(x^2 + y^2)(2y) \text{-------- (3)}$$

Dividing (2) by (3) $p\,y - q\,x = y^2 - x^2$

Hence the required partial differential equation is

$$p\,y - q\,x = y^2 - x^2$$

3. Form a partial differential equation by eliminating the arbitrary function from $z = f\left(x^2 - y^2\right)$

Sol. Given *equation is* $z = f\left(x^2 - y^2\right) \rightarrow$ (1)

Let $u = x^2 - y^2$, then $z = f(u) \rightarrow$ (2)

Differentiating (2) partially w.r.t. ‘x’ and ‘y’,

$$\frac{\partial z}{\partial x} = f^1(u).\frac{\partial u}{\partial x} = f^1(u).2x$$

$$\therefore p = f^1(u)2x \rightarrow \quad (3)$$

Similarly we get

$$q = -f^1(u)2y(4)$$

$$\therefore (3) \div (4), \text{ gives } \frac{p}{q} = \frac{x}{-y}$$

$$\therefore px + qy = 0$$

Hence the required partial differential equation is $\boxed{px + qy = 0}$

4. Form the partial differential equation by eliminating the arbitrary functions from

$xyz = f\left(x^2 + y^2 + z^2\right)$

Sol. Given *equation is* $xyz = f\left(x^2 + y^2 + z^2\right) \rightarrow$ (1)

Differentiating (1) partially w.r.t. x and y .

$$yz + xy.p = f^1\left(x^2 + y^2 + z^2\right).\left(2x + 2z.\frac{\partial z}{\partial x}\right)$$

$$yz + xyp = f^1\left(x^2 + y^2 + z^2\right).(2x + 2zp) \rightarrow \quad (2)$$

And $xz + xy.q = f^1\left(x^2 + y^2 + z^2\right).(2y + 2z.q) \rightarrow$ (3)

$\therefore (2) \div (3)$, gives

$$\frac{yz + xyp}{xz + xyq} = \frac{2x + 2zp}{2y + 2zq}$$

$$(yz + xyp)(y + zq) = (xz + xyq)(x + zp)$$

$$y^2z + z^2yq + xy^2p + xyzpq = x^2z + x^2zp + x^2yq + xyzpq$$

$$x\left(y^2 - z^2\right)p + y\left(z^2 - x^2\right)q = \left(x^2 - y^2\right)z$$

Hence the required partial differential equation is

$$\boxed{x\left(y^2 - z^2\right)p + y\left(z^2 - x^2\right)q = \left(x^2 - y^2\right)z}$$

5. **Form the partial differential equation by eliminating the arbitrary functions**

From $xyz = f(x + y + z)$

Sol.Given equations *is* $xyz = f(x + y + z)$---------(1)

Differentiating (1) partially w.r.t. 'x'

$$y(xp + z) = f'(x + y + z)(1 + p)\text{-------}(2)$$

Differentiating (1) partially w.r.t. 'x'

$$x(yq + z) = f'(x + y + z)(1 + q)\text{----------}(3)$$

Dividing (2) by (3) $\frac{y(xp+z)}{x(yq+z)} = \frac{1+p}{1+q}$

$$y(xp + z)(1 + q) = x(yq + z)(1 + p)$$

$$(xy - zx)p + (yz - xy)q = zx - yz$$

$$x(y-z)p + y(z-x)q = z(x-y)$$

Hence the required partial differential equation is

$$\boxed{x(y-z)p + y(z-x)q = z(x-y)}$$

6. Form the partial differential equation by eliminating the arbitrary function from $xy + yz + zx = f\left(\frac{z}{x+y}\right)$

Sol. Given equations *is* $xy + yz + zx = f\left(\frac{z}{x+y}\right) \rightarrow$ (1)

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$$y + y.p + z + x.p = f^1\left(\frac{z}{x+y}\right)\frac{\left[(x+y).p - z\right]}{(x+y)^2} \rightarrow \text{(2)}$$

$$x + z + yq + xq = f^1\left(\frac{z}{x+y}\right)\frac{\left[(x+y)q - z\right]}{(x+y)^2} \rightarrow \text{(3)}$$

Dividing (2) by (3), we get

$$\frac{(x+y)p + y + z}{(x+y)q + x + z} = \frac{(x+y)p - z}{(x+y)q - z}$$

$$\big((x+y)p + y + z\big)\big((x+y)q - z\big) = \big((x+y)q + x + z\big)\big((x+y)p - z\big)$$

$$(x+y)^2 pq + (y+z)(x+y)q - (x+y)pz - (y+z)z$$

$$= (x+y)^2 pq + (x+z)(x+y)p - (x+y)qz - (x+z)z$$

$$= (x+y)(x+2z)p - (x+y)(y+2z)q$$

$$= z(x-y)$$

$$(x+y)(x+2z)p - (x+y)(y+2z)q = z(x-y)$$

Hence the required partial differential equation is

$$\boxed{(x+y)(x+2z)p - (x+y)(y+2z)q = z(x-y)}$$

7.**Form the partial differential equation by eliminating the arbitrary function from** $z = f(x) + e^y.g(x)$

Sol. Given equations *is* $z = f(x) + e^y.g(x) \rightarrow$ (1)

Differentiating (1) partially w.r.t. 'x' and y, we get

$$\frac{\partial z}{\partial x} = f^1(x) + e^y.g^1(x) \text{ or } p = f^1(x) + e^y.g^1(x) \rightarrow \text{(2)}$$

$$q = e^y.g(x) \text{ or } \frac{\partial z}{\partial y} = e^y.g(x) \rightarrow \text{(3)}$$

Differentiating (3), partially w.r.t. 'y', we get

$$\frac{\partial^2 z}{\partial y^2} = e^y.g(x) = \frac{\partial z}{\partial y} \text{ [using (3)]}$$

$$\therefore \frac{\partial^2 z}{\partial y^2} - \frac{\partial z}{\partial y} = 0$$

$$\therefore t - q = 0$$

Hence the required partial differential equation is

$$\boxed{t - q = 0}$$

8.Form a partial differential equation by eliminating the arbitrary function $\boldsymbol{z = f(x^2 + y^2)}$

Sol. Given equations *is* $\boldsymbol{z = f(x^2 + y^2) \ldots\ldots (1)}$

Let $u = x^2 + y^2$,then $z = f(u) \rightarrow$ (2)

Differentiating (2) partially w.r.t. x and y,

$$\frac{\partial z}{\partial x} = f^1(u).\frac{\partial u}{\partial x} = f^1(u).2x$$

$$\therefore p = f^1(u)2x \rightarrow \text{(3)}$$

$$\frac{\partial z}{\partial y} = f^1(u).\frac{\partial u}{\partial y} = f^1(u).2y$$

$$\therefore q = f^1(u)2y \rightarrow \text{(4)}$$

$\therefore (3) \div (4)$, gives $\frac{p}{q} = \frac{f^1(u).2x}{f^1(u)2y} = \frac{x}{y}$

$\therefore py - qx = 0$

Hence the required partial differential equation is

$$\boxed{py - qx = 0}$$

10. Form a partial differential equation by eliminating the arbitrary function $\varphi(x^2 + y^2 + z^2, ax + by + cz) = 0$

Sol. Given function can be written as

$$x^2 + y^2 + z^2 = f(ax + by + cz) \ldots\ldots\ldots(1)$$

Differentiating (1) partially w.r.t. 'x' and 'y',we get

$$2x + 2zp = (a + cp)f^1(ax + by + cz) \ldots.(2)$$

$$2y + 2zq = (b + cq)f^1(ax + by + cz) \ldots.(3)$$

$\frac{(2)}{(3)}$ **implies** $\frac{x+zp}{y+zq} = \frac{(a+cp}{(b+cq)}$

Hence the required partial differential equation is

$$\boxed{\frac{x + zp}{y + zq} = \frac{(a + cp}{(b + cq)}}$$

SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

COMPLETE INTEGRAL

A solution in which the number of arbitrary constants is equal to the number of independent variables is called **complete integral** or **complete solution** of the given equation.

PARTICULAR INTEGRAL

A solution obtained by giving particular values to the arbitrary constants in the complete integral is called a **particular integral** or **particular solution**.

SINGULAR INTEGRAL

$Let\ f(x, y, z, p, q) = 0 \rightarrow$ (1) be the partial differential equation.

$Let\ \phi(x, y, z, a, b) = 0 \rightarrow$ (2)

Be the complete integral of (1). Where a and b are arbitrary constants.

Now find $\frac{\partial \phi}{\partial a} = 0 \rightarrow$ (3)

$\frac{\partial \phi}{\partial b} = 0 \rightarrow$ (4)

Eliminate a and b between the equations(2), (3) & (4) When it exists is called the **singular integral** of (1).

GENERAL INTEGRAL : In the complete integral (2). Assume that one of the constant is a function of the other i.e. b=f(a) Then (2), becomes

$$\phi(x, y, z, a, f(a)) = 0 \rightarrow (5)$$

Differentiating (5) partially w.r.t. 'a', we get

$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial f}.f^1(a) = 0 \rightarrow \quad (6)$$

Eliminate 'a' from (5) and (6), is called the **general integral** or **general solution** of (1).

LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

A differential equation involving partial derivatives p and q only and no higher order derivatives is called a **first order equation**. If p and q occur in the first degree, it is called a **linear partial differential equation of first order**, otherwise it is called a **non-linear partial differential equation of the first orde**r.

For example: $px + qy^2 = z$ is a **linear partial differential equation** and $p^2 + q^2 = 1$ is **non linear partial differential equation .**

LAGRANGE'S LINEAR PARTIAL DIFFERENTIAL EQUATION

A linear partial differential equation of order one involving a dependent variable z and two independent variables x and y of the form $Pp + Qq = R$

Where P, Q, R are functions of x, y, z is called **Lagrange's linear equation**.

Lagrange's auxiliary equations are

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}$$

WORKING RULE TO SOLVE LAGRANGE'S LINEAR EQUATION $Pp + Qq = R$

Step 1: Write down the auxiliary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Step 2 : Solve the auxiliary equations by the method of grouping or the method of multipliers or both to get two independent solutions u=a and v=b where a,b are arbitrary constants

Step 3: Then Q(u,v) =0 or u=f(v) is the general solution of the equation $Pp + Qq = R$

To solve $\boxed{\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}}$ …… (1)

(1)Method of grouping: In some problems, it is possible that two of the equations

$\frac{dx}{P} = \frac{dy}{Q}$ or $\frac{dy}{Q} = \frac{dz}{R}$ or $\frac{dx}{P} = \frac{dz}{R}$ are directly solvable to get solutions

u(x,y) = constant or

v(y,z)=constant or

w(x,z)=constant. These give the complete solutions of (1)

Sometimes one of them, say $\frac{dx}{P}=\frac{dy}{Q}$ may give rise to solution $u(x,y)=c_1$

From this we may express y, as a function of x. Using this in $\frac{dy}{Q}=\frac{dz}{R}$ and integrating we get

$v(y,z)=c_2$. *These two relations* $u=c_1, v=c_2$ *give the complete solution of* (1).

2. **Method of multipliers**: This is base on the following elementary result.

If $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=......=\frac{a_n}{b_n}$ then each ratio is equal to $\frac{l_1a_1+l_2a_2+....+l_na_n}{l_1b_1+l_2b_2+....+l_nb_n}$

Consider $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

If possible identity multipliers l, m, n, not necessarily constant, so that each ratio

$$=\frac{ldx+mdy+ndz}{lP+mQ+nR}$$

Where $L\,P+m\,Q+n\,R=0$ Then $l\,dx+m\,dy+n\,dz=0$

Integrating this we get $u(x,y,z)=c_1$.

Similarly we get another solution $v(x,y,z)=c_2$ independent of the earlier one.

We have the complete solution of (1) constituted *by* $u=c_1$ *and* $v=c_2$..

LINEAR PARTIAL DIFFERENTIAL EQUATIONS PROBLEMS

1. Solve $p\tan x+q\tan y=\tan z$

Sol. The given equation is $p\tan x+q\tan y=\tan z$ ....(1)

Comparing with $Pp+Qq=R$,

Where $P=\tan x, Q=\tan y, R=\tan z$

$\therefore$ The Auxiliary Equations are $\frac{dx}{\tan x}=\frac{dy}{\tan y}=\frac{dz}{\tan z}$

Taking the first two members, we have $\frac{dx}{\tan x}=\frac{dy}{\tan y}$

Integrating , we get

$$\log\sin x=\log\sin y+\log c_1$$

$$\log\frac{\sin x}{\sin y}=\log c_1 \;\; or \;\; \frac{\sin x}{\sin y}=c_1 \rightarrow \;(2)$$

Taking the last two members, we have $\frac{dy}{\tan y} = \frac{dt}{\tan z}$

Integrating, we get

$$log\ siny = log\ sinz + logc2$$

$$\log \frac{\sin y}{\sin z} = \log c_2 \ or \frac{\sin y}{\sin z} = c_2 \rightarrow \ (3)$$

From (2) and (3).

The General solution of (1) is

$$\phi(c_1, c_2) = 0$$

$$\boxed{i.e.\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0}$$

2.Find the general solution of $y^2zp + x^2zq = y^2x$

Sol. The given equation is $y^2zp + x^2zq = y^2x \rightarrow \ (1)$

Comparing $with\ Pp + Qq = R, we\ have$

$$P = y^2z, Q = x^2z,\ R = y^2x$$

$\therefore$ The auxiliary equations are $\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{y^2x}$

Taking the first two members, we have

$$\frac{dx}{y^2z} = \frac{dy}{x^2z} \Rightarrow \frac{dx}{y^2} = \frac{dy}{x^2} or\ x^2dx = y^2dy$$

Integrating, we get

$$\frac{x^3}{3} = y^3\!/_3 + c_1 \ or \ \frac{x^3}{3} - \frac{y^3}{3} = c_1 \rightarrow \ (2)$$

Taking the first and last two members, we have

$$\frac{dx}{y^2z} = \frac{dz}{y^2x} or\ xdx = zdz$$

Integrating ,we get

$$\frac{x^2}{2} = \frac{z^2}{2} + c_2 \ or \ \frac{x^2}{2} - \frac{z^2}{2} = c^2 \rightarrow \ (3)$$

From (2) and (3)

The General Solution of (1) is

$$\phi(c_1, c_2) = 0 \ i.e.$$

$$\boxed{\phi\left(\frac{x^3}{3} - \frac{y^3}{3}, \frac{x^2}{2} - \frac{z^2}{2}\right) = 0}$$

3.Solve $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

Sol. The given equation can be written as

$$\sqrt{x}p + \sqrt{y}q = \sqrt{z} \rightarrow (1)$$

Comparing with $Pp + Qq = R$, we have

$$P = \sqrt{x}, Q = \sqrt{y}, R = \sqrt{z}$$

$\therefore$ The auxiliary equations are $\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$

From the first two members, we have $\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}}$

Integrating,we get

$$2\sqrt{x} = 2\sqrt{y} + c_1 \ or \ 2\sqrt{x} - 2\sqrt{y} = c_1$$

$$\sqrt{x} - \sqrt{y} = a \rightarrow (2)$$

From the last two members, we have $\frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$

Integrating, we get

$$2\sqrt{y} = 2\sqrt{z} + c_2 \ or \ 2\sqrt{y} - 2\sqrt{z} = c_2$$

$$or \ \sqrt{y} - \sqrt{z} = b \rightarrow (3)$$

From (2) and (3).

The General Solution of (1) is

$$\phi(a, b) = 0$$

$$\boxed{\phi\left(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}\right) = 0}$$

4.Solve $x(y-z)p + y(z-x)q = z(x-y)$

Sol. The given equation can be written as

$$x(y-z)p+y(z-x)q=z(x-y) \rightarrow \ (1)$$

Comparing with $Pp+Qq=R$, we have

$$P=x(y-z), Q=y(z-x), R=z(x-y)$$

$\therefore$ The auxiliary equations are $\dfrac{dx}{x(y-z)}=\dfrac{dy}{y(z-x)}=\dfrac{dz}{z(x-y)}$

Using l=1,m=1, n=1 as multipliers, we get

$$\frac{dx}{x(y-z)}=\frac{dy}{y(z-x)}=\frac{dz}{z(x-y)}=\frac{dx+dy+dz}{0}$$

$$\therefore dx+dy+dz=0 \quad [\therefore x(y-z)+y(z-x)+z(x-y)=0]$$

Integrating, we get

$$x+y+z=a \rightarrow (2)$$

Again using $l=\dfrac{1}{x}, m=\dfrac{1}{y}, n=\dfrac{1}{z}$ as multipliers, we get

Each fraction $=\dfrac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$ =k(say)

$$\therefore \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$$

Integrating, we get

$$logx+logy+logz=logb. or\ xyz=b \ldots\ldots(3)$$

From (2) and (3).

The General Solution of (1) is

$$\phi(a,b)=0$$

$$\boxed{\phi(x+y+z, xyz)=0}$$

5.Solve $x^2(y-z)p+y^2(z-x)q=z^2(x-y)$

Sol. The given equation can be written as

$$x^2(y-z)p+y^2(z-x)q=z^2(x-y) \rightarrow (1)$$

Comparing with $Pp+Qq=R$, we have

$$P = x^2(y-z), Q = y^2(z-x), R = z^2(x-y)$$

$\therefore$ The auxiliary equations are $\dfrac{dx}{x^2(y-z)} = \dfrac{dy}{y^2(z-x)} = \dfrac{dz}{z^2(x-y)}$

Using $l = \dfrac{1}{x^2}, m = \dfrac{1}{y^2}, n = \dfrac{1}{z^2}$ as multipliers, we get

Each fraction $= \dfrac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$ =k(say)

$$\therefore \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

Integrating ,we get

$$-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = a \quad or \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_1 \rightarrow (2)$$

Again using $l = \dfrac{1}{x}, m = \dfrac{1}{y}, n = \dfrac{1}{z}$ as multipliers, we get

Each fraction $= \dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$ =k(say)

$$\therefore \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating ,we get

$$\log x + \log y + \log z = \log c_2$$

$$or\ xyz = c_2 \rightarrow (3)$$

From (2) and (3),

The General Solution of (1) is

$$\boxed{\phi\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right) = 0}$$

6.Solve $(mz - ny)p + (nx - lz)q = ly - mx$

Sol. The given equation can be written as

$$(mz - ny)p + (nx - lz)q = ly - mx \rightarrow (1)$$

Comparing with $Pp + Qq = R$, we have

$$P = mz - ny, Q = nx - lz, R = ly - mx$$

$\therefore$ The auxiliary equations are

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

Using l=x, m=y, n=z as multipliers, we get

$$\text{Each fraction } = \frac{xdx+ydy+zdz}{0}$$

$$\therefore xdx + ydy + zdz = 0$$

Integrating,we get

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} = a \; or \; x^2+y^2+z^2 = c_1 \rightarrow (2)$$

Again using l, m, n as multipliers, we get

$$\text{Each fraction } = \frac{ldx+mdy+nz}{0} = \text{k(say)}$$

$$\therefore ldx+mdy+ndz=0$$

Integrating,we get

$$lx+my+nz = c_2 \rightarrow (3)$$

From (2) and (3),

The General Solution of (1) is

$$\boxed{\phi\left(x^2+y^2+z^2, lx+my+nz\right) = 0}$$

7.Solve $xp - yq = y^2 - x^2$

Sol. The given equation can be written as

$$xp - yq = y^2 - x^2$$

Comparing with $Pp + Qq = R$, we have

$$P = x, Q = y, R = y^2 - x^2$$

$\therefore$ The auxiliary equations are $\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{y^2-x^2}$

From the first two members, $\frac{dx}{x} = \frac{dy}{-y}$

Integrating,we get

$$\log x + \log y = \log c_1 \; or \; xy = c_1 \rightarrow (1)$$

Using l=x, m=y, n=1 as multipliers, we get

Each fraction $= \dfrac{xdx + ydy + dz}{0}$

$$\therefore xdx + ydy + dz = 0$$

Integrating,we get

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 + z = c \, or \, x^2 + y^2 + 2z = c_2 \rightarrow \quad (2)$$

From (1) and (2) ,

The General Solution is

$$\boxed{\phi\left(xy, x^2 + y^2 + 2z\right) = 0}$$

8. Find the integral surface of $x\left(y^2 + z\right)p - y\left(x^2 + z\right)q = \left(x^2 - y^2\right)z$

Which contains the straight line x+y=0, z=1

Sol. The given equation can be written as

$$x\left(y^2 + z\right)p - y\left(x^2 + z\right)q = \left(x^2 - y^2\right)z \; \text{........(1)}$$

Comparing with $Pp + Qq = R$, we have

$$P = x\left(y^2 + z\right), Q = -y\left(x^2 + z\right), R = \left(x^2 - y^2\right)z$$

$\therefore$ The auxiliary equations are

$$\frac{dx}{x\left(y^2 + z\right)} = \frac{dy}{-y\left(x^2 + z\right)} = \frac{dz}{\left(x^2 - y^2\right)z}$$

Using $l = \frac{1}{x}, m = \frac{1}{y}, n = \frac{1}{z}$ as multipliers, we get

$$\text{Each fraction} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\therefore \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating,we get

$$\log x + \log y + \log z = \log a$$

$$xyz = a \rightarrow \quad (2)$$

Again using l=x, m=y, n=-1 as multipliers, we get

$\therefore$ Each fraction $= \dfrac{xdx + ydy - dz}{0}$ =k(say)

$$\therefore xdx + ydy - dz = 0$$

Integrating,we get

$$\frac{x^2}{2} + \frac{y^2}{2} - z = c \; or \; x^2 + y^2 - 2z = b \rightarrow (3)$$

Given that z=1, using this (2) and (3), we get

$$xy = a \; and \; x^2 + y^2 - 2 = b$$

Now

b+2a= $x^2 + y^2 - 2 + 2xy = (x+y)^2 - 2$ =0-2 $[\therefore x + y = 0]$=-2

$$\therefore 2a + b + 2 = 0$$

Hence the required surface is

$$\boxed{x^2 + y^2 - 2z + 2xyz + 2 = 0}$$

9.Solve $px + qy = z$

Sol. The given equation can be written as

$px + qy = z$ is a Lagrange's linear equation

The Auxillary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

By Consider first group, we get

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\log x = \log y + \log c1$$

$$c_1 = \frac{x}{y} \ldots..(1)$$

By Consider second group, we get

$$\int \frac{dz}{z} = \int \frac{dy}{y}$$

$$\log y = \log z + \log c2$$

$$c_2 = \frac{y}{z} \ldots..(2)$$

The General Solution is

$$\therefore f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$$

10.Solve $(x^2 - y^2 - yz)p + (x^2 - y^2 - xz)q = z(x - y)$

Sol. The given equation can be written as

$$(x^2 - y^2 - yz)p + (x^2 - y^2 - xz)q = z(x - y)\text{.........(1)}$$

The auxiliary equations of (1) are

$$\frac{dx}{(x^2-y^2-yz)} = \frac{dy}{(x^2-y^2-xz)} = \frac{dz}{z(x-y)}$$

Taking 1,-1-1 multipliers,we get

$$\frac{dx-dy-dz}{(x^2-y^2-yz-x^2+y^2+xz-xz+yz)} = \frac{dx}{(x^2-y^2-yz)}$$

$$dx - dy - dz = 0$$

Integrating,we get

$$x - y - z = c_1\text{.........(2)}$$

Taking $x, -y, 0$ as multipliers,we get

$$\frac{xdx-ydy}{(x^3-xy^2-xyz-yx^2+y^3+xyz)} = \frac{dz}{z(x-y)}$$

$$\frac{xdx-ydy}{(x^2-y^2)(x-y)} = \frac{dz}{z(x-y)}$$

Integrating,we get

$$\frac{1}{2}\log(x^2 - y^2) = \log z$$

$$\frac{x^2-y^2}{z^2} = c_2 \text{ ... ... (2)}$$

$$\therefore \textit{Complete solution of given pde is } \varphi\left(x - y - z, \frac{x^2 - y^2}{z^2}\right) = 0$$

11.Solve $x(y^2 - z^2)p - y(x^2 + z^2)q = z(x^2 + y^2)$

Sol. The given equation can be written as

$$x(y^2 - z^2)p - y(x^2 + z^2)q = z(x^2 + y^2)$$

The auxiliary equations are

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(x^2 + z^2)} = \frac{dz}{z(x^2 + y^2)}$$

Taking x, y, z, multipliers, we get

$$\frac{xdx+ydy+zdz}{(x^2y^2-x^2z^2-y^2x^2-z^2y^2+x^2z^2+y^2z^2)}=\frac{dx}{x(y^2-z^2)}$$

$$xdx + ydy + zdz = 0$$

$$x^2 + y^2 + z^2 = c_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Taking $-\frac{1}{x}, \frac{1}{y}\frac{1}{z}$, multipliers, we get

$$-\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating, we get

$$\frac{yz}{x} = c_2 \ldots\ldots (2)$$

From (1),(2),

$\therefore$ *Complete solution of given pde is* $\varphi\left(\frac{yz}{x}, x^2 + y^2 + z^2\right) = 0$

12.Solve $(y^2)p - xyq = x(z - 2y)$

Sol. The given equation can be written as

$$(y^2)p - xyq = x(z - 2y)\ldots\ldots\ldots(1)$$

Comparing with $Pp + Qq = R$, we have

The auxiliary equations are

$$\therefore \frac{dx}{y^2} = \frac{dy}{-yx} = \frac{dz}{x(z-2y)}$$

From the first two members, we have Type equation here.

$$\frac{dx}{y} = \frac{dy}{-x}$$

Integrating,we get

$$x^2 + y^2 = c_1 \ldots\ldots(2)$$

From the last two members, we have

$$\frac{dy}{-y} = \frac{dz}{(z-2y)}$$

$$-ydz = zdy - 2ydy$$

$$d(yz) - 2ydy = 0$$

$$yz - y^2 = c_2 \text{ ....(3)}$$

From (2) and (3).

The General Solution of (1) is

$$i.e., \ \emptyset(yz - y^2, x^2 + y^2) = 0$$

13.Solve $(y+z)p + (z+x)q = (x+y)$

Sol.The given equation can be written as

$$(y+z)p + (z+x)q = (x+y) \text{ ....(1)}$$

Comparing with $Pp + Qq = R$, we have

The auxiliary equations are

∴

$$\therefore \frac{dx}{(y+z)} = \frac{dy}{(z+x)} = \frac{dz}{(x+y)}$$

Taking 1,1,1 and 1,-1,0 and 0,1,-1 as multipliers ,

we have $\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{(y-x)} = \frac{dy-dz}{(z-y)}$

From the last two members, we have

$$\frac{dx-dy}{(y-x)} = \frac{dy-dz}{(z-y)}$$

Integrating, we get

$$\log\frac{(y-x)}{(z-y)} = \log C_2$$

$$\frac{(y-x)}{(z-y)} = c_2 \text{.....(2)}$$

From the first two members, we have

$$\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{(y-x)}$$

Integrating, we get

$$\frac{1}{2}\log(x+y+z) = \log(y-x) + \log c_1$$

$$(x+y+z)(y-x)^2 = C_1 \text{..........(3)}$$

From (2) and (1).

The General Solution of given pde is

$$i.e., \emptyset\left(\frac{(y-x)}{(z-y)}, (x+y+z)(y-x)^2\right)=0$$

14.Solve $x^2p - y^2q = z(x-y)$

Sol.The given equation can be written as

$$x^2p - y^2q = z(x-y)$$

Comparing with $Pp + Qq = R$, we have

The auxiliary equations are

$\therefore$

$$\frac{dx}{x^2} = \frac{dy}{-y^2} = \frac{dz}{(z(x-y)}$$

From the first two members, we have

$$\frac{dx}{x^2} = \frac{dy}{-y^2}$$

Integrating,we get

$$\frac{1}{x} + \frac{1}{y} = c_1 \text{.....(1)}$$

Taking 1,1,0 as multipliers,we get

$$\frac{dx+dy}{x^2-y^2} = \frac{dz}{(z(x-y)}$$

$$\frac{dx+dy}{(x+y)(x-y)} = \frac{dz}{(z(x-y)}$$

$$\frac{dx+dy}{(x+y)} = \frac{dz}{z}$$

Integrating,we get

$$\frac{x+y}{z} = c_2 \quad \text{....(2)}$$

From (2) and (1).

The General Solution is

$$i.e., \ \emptyset\left(\frac{x+y}{z}, \frac{1}{x}+\frac{1}{y}\right)=0$$

15.Solve $(x^2 - yz)p + (y^2 - xz)q = (z^2 - xy)$

Sol.The given equation can be written as

$$(x^2 - yz)p + (y^2 - xz)q = (z^2 - xy)$$

The auxiliary equations are

$$\frac{dx}{(x^2 - yz)} = \frac{dy}{(y^2 - xz)} = \frac{dz}{(z^2 - xy)}$$

Taking 1,-1,0 and 0,-1,-1 as multipliers,we get

$$\frac{dx-dy}{(x^2-yz)-(y^2-xz)} \text{ and also } \frac{dy-dz}{\left((-z^2+yx)+(y^2-xz)\right)}$$

$$\therefore \frac{dx-dy}{(x^2-yz)-(y^2-xz)} = \frac{dy-dz}{\left((-z^2+yx)+(y^2-xz)\right)}$$

$$\frac{d(x-y)}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z))}$$

$$\textit{solving it, we get}$$

$$\frac{(x-y)}{y-z}=c_1\ldots\ldots(1)$$

Taking x, y, z and 1,1,1 *as* multipliers,we get

$$\frac{(xdx+ydy+zdz)}{x^3+y^3+z^3-3xyz} = \frac{(dx+dy+dz)}{x^2+y^2+z^2-xy-yz-zx}$$

$$\frac{(xdx+ydy+zdz)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{(dx+dy+dz)}{x^2+y^2+z^2-xy-yz-zx}$$

$$(x + y + z)(dx + dy + dz) = (xdx + ydy + zdz)$$

$$(x + y + z)d(x + y + z) = (xdx + ydy + zdz)$$

Integrating,we get

$$\frac{(x + y + z)^2}{2} = \frac{x^2 + y^2 + z^2}{2} + c$$

$$\therefore (x + y + z)^2 = x^2 + y^2 + z^2 + c_2$$

$$xy + yz + zx = c_2 \ldots\ldots(2)$$

$$\therefore \text{ Complete solution of given pde is } \varphi\left(xy + yz + zx, \frac{(x-y)}{y-z}\right) = 0$$

NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

A partial differential equation which involves first order partial derivatives p *and* q with degree higher than one and the products *of* p *and* q *is* called a **non-linear partial differential equations**.

CHARPIT'S METHOD FOR FINDING THE COMPLETE INTEGRAL OF NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

In this method give partial differential Equation of the form $f(x, y, z, p, q) = 0$ to find another relation of the form $\varphi(x, y, z, p, q) = 0$ which is compatible with *the*

$$f(x, y, z, p, q) = 0 \text{ then}$$

we solve for p, q and substitute these values in the relation

$$dz = p\,dx + q\,dy.$$

Which on integration gives the required solution of

$$f(x, y, z, p, q) = 0$$

CHARPIT'S EQUATION

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{[f_x + pf_z]} = \frac{dq}{[f_y + qf_z]}$$

PROBLEMS

1.Solve $\boldsymbol{px + qy = pq}$

Sol. Given **equation is** $\boldsymbol{f = px + qy - pq}$

The auxillary equations are

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{[f_x + pf_z]} = \frac{dq}{[f_y + qf_z]}$$

Here $f_p = x;\ f_q = y;\ f_x = p\ ;\ f_y = q\ ;\ f_z = 0$

Substituting them in above ,we get

$$\frac{dx}{-x} = \frac{dy}{-y} = \frac{dz}{px + qy} = \frac{dp}{p} = \frac{dq}{q}$$

By considering first and last groups,

$$\frac{dx}{-x} = \frac{dy}{-y}$$

Integrating,we get

$$\log x = \log y + \log c$$

$$\therefore y = cx \ldots..(1)$$

$$\frac{dp}{p} = \frac{dq}{q}$$

Integrating,we get

$$\log p = \log q + \log a$$

$$\therefore p = aq \ldots..(2)$$

Substitute (1),(2) in given equation ,we get

$$p = ax + y\,; q = \frac{ax + y}{a}$$

The General solution is

$$dz = p\,dx + q\,dy$$

$$dz = (ax + y)\,dx + \frac{ax + y}{a}\,dy$$

Integrating it we get required solution as

$$z = \frac{1}{a}(ax + y)^2 + c$$

2.Solve $\boldsymbol{z^2 = pqxy}$

Sol. Given equation is $f = \boldsymbol{z^2 - pqxy}$

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{[f_x + pf_z]} = \frac{dq}{[f_y + qf_z]}$$

Here $f_p = -qxy;\ f_q = -pxy;\ f_x = -qxy\,;\ f_y = -pxy\,;\ f_z = 2z$

Substituting them in above ,we get

$$\frac{dx}{qxy} = \frac{dy}{pxy} = \frac{dz}{2pqxy} = \frac{dp}{-pqy + 2pz} = \frac{dq}{-pqx + 2qz}$$

Considering

$$\frac{dx}{qxy} = \frac{dp}{-pqy+2pz} \text{ and } \frac{dy}{pxy} = \frac{dq}{-pqx+2qz}$$

Taking

p, x *as multipliers for first and* q, y *for second group, then equating them we get*

$$\frac{xdp+pdx}{-xpqy+2xpz+pqxy} = \frac{ydq+qdy}{-ypqx+2qz+pqxy}$$

Solving ,

$$\frac{d(px)}{px} = \frac{d(qy)}{qy}$$

Integrating,we get

$$\log px = \log qy + \log c$$

$$px = qyc$$

Substituting in given pde,we get

$$z^2 = q^2y^2c$$

$$\therefore q = \frac{z}{y\sqrt{c}} \; ; p = \frac{z}{x}\sqrt{c}$$

The General Solution is

$$dz = p\,dx + q\,dy$$

$$dz = \frac{z}{x}\sqrt{c}dx + \frac{z}{y\sqrt{c}}dy$$

$$\int \frac{dz}{z} = \int \frac{1}{x}\sqrt{c}dx + \int \frac{1}{y\sqrt{c}}dy$$

Integrating,we get,

$\therefore$ Required solution is $z = x^{\sqrt{c}} y^{\frac{1}{\sqrt{c}}} a$

Now let us start solving some standard forms of First order partial differential equations by using

CHARPIT'S METHOD.

STANDARD FORM I:

Equation Of The Form f(p,q)=0 :

Note: equations containing p and q only.

Given partial differential equation is $\boldsymbol{f(p,q) = 0} \dots\dots (\mathbf{1})$

The auxillary equations are

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{[f_x + pf_z]} = \frac{dq}{[f_y + qf_z]}$$

Here $f_x = 0\,;\; f_y = 0\,;\; f_z = 0$

Substituting above and considering last group, we get

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{0} = \frac{dq}{0}$$

$$\therefore \boldsymbol{dp = 0}, integrating\ we\ get\ \boldsymbol{p = a}$$

Put $\boldsymbol{p = a\ in\ (1), then\ we\ get\ q\ value\ in\ terms\ of\ a\ , say\ \varphi(a).}$

But we have

$$\boldsymbol{dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy}$$

$$\boldsymbol{dz = p\,dx + q\,dy}$$

$$\boldsymbol{dz = a\,dx + \varphi(a)dy \ldots\ldots (2)}$$

Integrating (2),we get required complete solution of (1) is $\boxed{z = ax + \phi(a)y + c}$

Which contains two arbitrary constants a and c.

PROCEDURE:

Given partial differential equation is $\boldsymbol{f(p,q) = 0} \ldots \ldots \boldsymbol{(1)}$

STEP1: Put $\boldsymbol{p = a}\ in\ (1), then\ we\ get\ q\ value\ in\ terms\ of\ a$ ..then we can obtain 'p' value.

$\boldsymbol{STEP2}: Sub\ p, q\ values\ in\ \boldsymbol{dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy}$

$$\boldsymbol{i.e\ dz = p\,dx + q\,dy}$$

STEP3:Integrating it ,we get required complete solution of (1) .

PROBLEMS

1. Solve $\boldsymbol{pq = k}$, where k is a constant.

Sol. Given equation is $pq = k \ldots .(1)$

Since equation (1) is of the form $f(p,q) = 0$

Put **p=a** in (1),we get $\boldsymbol{q = \frac{k}{a}}$

The General Solution is

$$\boldsymbol{dz = p\,dx + q\,dy}$$

$$\boldsymbol{dz = a\,dx + \frac{k}{a} dy}$$

Integrating,we get ,

$$z = ax + \frac{k}{a} y + c$$

which contains two arbitrary constants a and c.

2.Solve $p^2 + q^2 = npq$

Sol. Given equation is $p^2 + q^2 = npq$ ………..(1)

Since equation is (1) is of the form $f(p,q) = 0$

Put p=a in (1),then we get $q = \frac{a}{2}\left[n \pm \sqrt{n^2 - 4}\right]$

The General Solution is

$$dz = p\,dx + q\,dy$$

$$dz = a\,dx + \frac{a}{2}\left[n \pm \sqrt{n^2 - 4}\right] dy$$

Integrating,we get ,

$$dz = a\int dx + \frac{a}{2}\left[n \pm \sqrt{n^2 - 4}\right]\int dy$$

$$z = ax + \frac{a}{2}\left[n \pm \sqrt{n^2 - 4}\right] y + c$$

This is the complete integral of (1), which contains. two arbitrary constants a and c.

3.Find the complete integral of $p^2 + q^2 = m^2$

Sol. Given equation is $p^2 + q^2 = m^2$ ……….(1)

Since equation (1) is of the form $f(p,q) = 0$

Put $p = a\ in\ (1), we\ get\ q = \sqrt{m^2 - a^2}$

The General Solution is

$$dz = p\,dx + q\,dy \quad \text{…….(2)}$$

Put the values of p,q in (2), we get

$$z = ax + \left(\sqrt{m^2 - a^2}\right) y + c$$

Which is the complete integral of (1).

STANDARD FORM II :

Equation Of The Form $f(p,q,z) = 0$ (i.e., not containing x and y)

PROCEDURE

Given partial differential equation is $\boldsymbol{f(p,q,z)=0}$ ... ... $\mathbf{(1)}$

STEP1: Put $\boldsymbol{p=aq}$ $in\ (1), then\ we\ get\ q\ value\ in\ terms\ of\ a\ ,z$ .then

STEP2: *Sub p, q values in* $\boldsymbol{dz=\frac{\partial z}{\partial x}\,dx+\frac{\partial z}{\partial y}\,dy}$

$$\boldsymbol{i.e\ dz\ =\ p\,dx+q\,dy}$$

STEP3: Integrating it ,we get required complete solution of (1) .

PROBLEMS

Solve the following partial differential equations

1. **(a)**$\boldsymbol{z=p^2+q^2}$ **(b)** $\boldsymbol{p^2z^2+q^2=p^2q}$ **(c)** $\boldsymbol{zpq=p+q}$

Sol.a). Given equation is $\boldsymbol{z=p^2+q^2}$ ... . (1)

Since (1) is of the form $f(z,p,q)=0$

Put $\boldsymbol{p=aq}$ $in\ (1), then\ we\ get\ q=\sqrt{\frac{z}{1+a^2}}$

$$\therefore p=a\sqrt{\frac{z}{1+a^2}}$$

Putting the values $of\ p\ and\ q\ in$ $\boldsymbol{dz\ =\ p\,dx+q\,dy}$, we get

$$\boldsymbol{\frac{1}{\sqrt{z}}dz\ =\frac{1}{\sqrt{1+a^2}}\,(a\,dx+\ dy)}\ ,$$

Integrating ,we get

$$\boldsymbol{\int\frac{1}{\sqrt{z}}dz\ =\frac{1}{\sqrt{1+a^2}}\int(a\,dx+\ dy)}$$

$\therefore\ \boldsymbol{2\sqrt{z}=\frac{1}{\sqrt{1+a^2}}(ax+y)}$ is the required solution of (1)

b)Given equation is $p^2z^2+q^2=p^2q\rightarrow$ (1)

Since (1) is of the form $f(z,p,q)=0$

Put $\boldsymbol{p=aq}$ $in\ (1), then\ we\ get\ q=\frac{(a^2z^2+1)}{a^2}$

$$\therefore p=\frac{(a^2z^2+1)}{a}$$

Putting the values of p and q in $\boldsymbol{dz\ =\ p\,dx+q\,dy}$,we get

$$\frac{dz}{(a^2z^2+1)} = \frac{1}{a^2}(adx + dy)$$

Integrating,we get

$$\int \frac{dz}{(a^2z^2+1)} = \frac{1}{a^2}\int(adx + dy)$$

$\therefore$ a $\tan^{-1}(az) = ax + y + c$ is the required complete solution of (1)

c) Given equation is $zpq = p + q$…..(1)

Since (1) is of the form $f(z, p, q) = 0$

Put $\boldsymbol{p} = \boldsymbol{aq}$ *in* (1), *then we get* $q = \frac{a+1}{az}$

$$\therefore p = \frac{a+1}{z}$$

Putting the values *of p and q in* $\boldsymbol{dz} = \boldsymbol{p\,dx} + \boldsymbol{q\,dy}$, we get

$$zdz = \frac{a+1}{a}(a\,dx + dy) \;,$$

Integrating ,we get

$$\int z\,dz = \frac{a+1}{a}\int(a\,dx + dy)$$

$\therefore \frac{az^2}{2(a+1)} = ax + y + c$ is the required solution of (1)

STANDARD FORM III :

Equation of the form $f_1(x, p) = f_2(y, q)$ **i.e. Equations not involving z and the terms containing x and p can be separated from those containing y and q.**

We assume that these two functions should be equal to a constant say k.

$$\therefore f_1(x, p) = f_2(y, q) = k$$

Solve for p and q from the resulting equations

$$\therefore f_1(x, p) = k \text{ and } f_2(y, q) = k$$

Solve for p and q, we obtain

$$p = F_1(x, k) \text{ and } q = F_2(y, k)$$

Since z is a function of x and y

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \text{ [By total differentiation]}$$

$$\boldsymbol{dz = p\,dx + q\,dy}$$

$$\therefore dz = F_1(x,k)dx + F_2(y,k)dy$$

Integrating on both sides

$z = \int F_1(x,k)dx + \int F_2(y,k)dy + c$ **is the complete solution of given equation .**

PROBLEMS

1. Solve the following partial differential equations

a). $p^2 + q^2 = x + y$ b). $xp - yq = y^2 - x^2$ c). $\left(\frac{p}{2}+x\right)^2 + \left(\frac{q}{2}+y\right)^2 = 1$

Sol .

a). Given equation is $p^2 + q^2 = x + y$ ....(1)

Separating p and x $from$ q and y, the given equation can be written as

$$p^2 - x = -q^2 + y$$

$$\text{Let } p^2 - x = -q^2 + y = k \text{ (constant)}$$

$$\therefore p^2 - x = k \text{ and } -q^2 + y = k$$

$$\Rightarrow p^2 = k + x \text{ and } q^2 = y - k$$

$$\therefore p = \sqrt{k+x} \text{ and } q = \sqrt{y-k}$$

$$\text{Since } dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$$

$$\therefore dz = \sqrt{k+x}\,dx + \sqrt{y-k}\,dy$$

Integrating on both sides

$$z = \int (k+x)^{\frac{1}{2}}dx + \int (y-k)^{\frac{1}{2}}dy + c$$

$\therefore z = \frac{2}{3}(k+x)^{\frac{3}{2}} + \frac{2}{3}(y-k)^{\frac{3}{2}} + c$**, is the complete solution of (1)**

b). Given equation is $xp - yq = y^2 - x^2 \rightarrow (1)$

Separating p and x from q and y.

The given equation can be written as.

$$xp + x^2 = yq + y^2$$

Let $xp + x^2 = yq + y^2 = k$ (arbitrary constant)

$$\therefore xp + x^2 = k \text{ and } yq + y^2 = k$$

$$\Rightarrow p = \frac{k - x^2}{x} \text{ and } q = \frac{k - y^2}{y}$$

We have $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$

$$\therefore dz = \left(\frac{k}{x} - x\right)dx + \left(\frac{k}{y} - y\right)dy$$

Integrating on both sides

$$z = \int\left(\frac{k}{x} - x\right)dx + \int\left(\frac{k}{y} - y\right)dy + c$$

$$= k\log x - \frac{x^2}{2} + k\log y - \frac{y^2}{2} + c$$

$\therefore z = k\log(xy) - \frac{1}{2}\left(x^2 + y^2\right) + c$ **, is the complete integral of (1)**

c). Given equation is $\left(\frac{p}{2} + x\right)^2 + \left(\frac{q}{2} + y\right)^2 = 1 \rightarrow$ **(1)**

Separating p *and* x *from* q *and* y, the given equation can be written as.

$$\left(\frac{p}{2} + x\right)^2 = 1 - \left(\frac{q}{2} + y\right)^2$$

Let $\left(\frac{p}{2} + x\right)^2 = 1 - \left(\frac{q}{2} + y\right)^2 = k^2$ (arbitrary constant)

$$\therefore \left(\frac{p}{2} + x\right)^2 = k^2 \text{ and } 1 - \left(\frac{q}{2} + y\right)^2 = k^2$$

$$\Rightarrow \frac{p}{2} + x = k \text{ and } \left(\frac{q}{2} + y\right)^2 = 1 - k^2 \text{ or } \frac{q}{2} + y = \sqrt{1 - k^2}$$

$$\Rightarrow p = 2(k-x) \text{ and } q = 2\left[\sqrt{1-k^2} - y\right]$$

We have $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$

$$\therefore dz = 2(k-x)dx + 2\left[\sqrt{1-k^2} - y\right]dy$$

Integrating on both sides

$$z = 2\int (k-x)dx + 2\int\left[\sqrt{1-k^2} - y\right]dy + c$$

$$z = 2(kx - \frac{x^2}{2}) + 2\left[\left(\sqrt{1-k^2}\right)y - \frac{y^2}{2}\right] + c$$

$\therefore z = 2kx - x^2 + 2\left(\sqrt{1-k^2}\right)y - y^2 + c$ **is the complete solution of (1)**

2.**Solve** $\boldsymbol{p - x^2 = q + y^2}$

Sol.

Let $\boldsymbol{p - x^2 = q + y^2 = k^2(say)}$

Then $\boldsymbol{p - x^2 = k^2}$ **and** $\boldsymbol{q + y^2 = k^2}$

$$\therefore \boldsymbol{p = k^2 + x^2 \text{ and } q = k^2 + y^2}$$

But we have

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$$

$$dz = (\boldsymbol{k^2 + x^2})dx + (\boldsymbol{k^2 + y^2})dy$$

Integrating,we get

$\boldsymbol{z = \frac{x^3}{3} + k^2x + k^2y + \frac{y^3}{3}}$ **+c is the required complete solution.**

3. Solve $\boldsymbol{q^2 - p = y - x}$

Sol . Let $p - x = q^2\text{-}y = k(say)$

Then $p = k + x \text{ and } q = \sqrt{k + y}$

But

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$$

$$\therefore dz = (\boldsymbol{k}+\boldsymbol{x})dx + (\sqrt{k+y})dy$$

Integrating,we get

$$\boldsymbol{z} = \frac{x^2}{2} + \boldsymbol{kx} + \frac{2}{3}(\boldsymbol{k}+\boldsymbol{y})^{\frac{3}{2}}$$ **+C is the required complete solution**.

4.Solve $q = px + p^2$

Sol. Let $q = px + p^2 = k(say)$

Then we get

$$p^2 + px - k = 0 \ and \ q = k$$

Solving,we get

$$p = \frac{-x \pm \sqrt{(x^2+4k}}{2} \quad \text{and } q = k$$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$$

$$\therefore dz = \left(\frac{-x \pm \sqrt{(x^2+4k}}{2}\right)dx + k\,dy$$

Integrating,we get

$$z = -\frac{x^2}{4} + \frac{1}{2}\left[\frac{x}{2}\sqrt{(x^2+4k} + 2k\sinh^{-1}\left(\frac{x}{2\sqrt{k}}\right)\right.$$ **+ky+C** is the required complete solution.

STANDARD FORM IV: $Z = px + qy + f(p,q)$:

An equation analogous to the clairaut's equation it is complete solution is

$$Z = ax + by + f(a,b)$$

Which is Obtained by writing $a \ for \ p \ and \ b \ for \ q$ The differential equation which satisfies some specified conditions known as the boundary conditions. The differential equation together with these boundary conditions, constitute a **boundary value problem .**

PROBLEMS

1. **Solve $z = px + qy + pq$**

Sol. Given equation is $z = px + qy + pq \ldots .(1)$

Since (1) is of the form $z = px + qy + f(p, q)$.

Hence the complete solution of (1) is given by

$$z = ax + by + ab \ldots \ldots .(2)$$

For singular solution, differentiating (2) partially w.r.t. a and b, we get

$$0 = x + b \ldots .(3) \quad and$$

$$0 = y + a \ldots \ldots \ldots \ldots (4)$$

Eliminating a, b between (2), (3) and (4), we get

$z = -xy - xy + xy = -xy$ is the Singular Solution.

2. **Find the solution of $(p+q)(z-px-qy)=1$**

Sol. The given equation can be written as

$$z - px - qy = \frac{1}{p+q}$$

$$\therefore z = px + qy + \frac{1}{p+q} \to (1)$$

Hence the complete solution of (1) is given by

$$z = ax + by + \frac{1}{a+b}$$

3.Solve $pqz = p^2(qx + p^2) + q^2(py + q^2)$

Sol. The given equation can be written as

$$pqz = p^2 q\left(x + \frac{p^2}{q}\right) + q^2 p\left(y + \frac{q^2}{p}\right)$$

$$\therefore z = p\left(x + \frac{p^2}{q}\right) + q\left(y + \frac{q^2}{p}\right)$$

$$\therefore z = px + qy + \left(\frac{p^3}{q} + \frac{q^3}{p}\right) \to (1)$$

Since it is in the form $z = px + qy + f(p,q)$

Hence the complete solution of (1) is given by

$$\boxed{z = ax + by + \frac{a^3}{b} + \frac{b^3}{a}}$$

4. Solve $z = px + qy + pq + q^2$

Sol. Given equation is $z = px + qy + pq + q^2 \ldots\ldots\ldots\ldots\ldots(1)$

Since (1) is of the form $z = px + qy + f(p,q)$.

Hence the complete solution of (1) is given by

$$z = ax + by + ab + b^2 \ldots\ldots(2)$$

For singular solution, differentiating (2) partially w.r.t. a and b, we get

$$\frac{\partial z}{\partial a} = 0, \frac{\partial z}{\partial b} = 0,$$

Implies that

$$0 = x + b \ldots.(3) \quad and$$

$$0 = y + a + 2b \ldots\ldots\ldots\ldots(4)$$

Eliminating a, b between (2), (3) and (4), we get

$$z = x(2x - y) - xy - (2x - y)x + x^2$$

$$\boxed{\therefore z = x^2 \text{ is the singular solution}}$$

EQUATIONS REDUCIBLE TO STANDARD FORMS

EQUATIONS OF THE FORM $F(x^m p, y^n q) = 0$ ($where\ m\ and\ n\ are\ constants$)

The above form of the equation of the type can be transformed to an equation of the form f(p,q)=0

By substitutions given below.

Case (i):- When $m \neq 1\ and\ n \neq 1$

Put $X = x^{1-m}\ and\ Y = y^{1-n}\ then\ p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} = P(1-m)x^{-m}\ whereP = \frac{\partial z}{\partial X}$

$$x^m p = P(1-m) \quad and\ q = \frac{\partial z}{\partial y} = \frac{\partial Z}{\partial Y}\frac{\partial Y}{\partial y} = Q(1-n)y^{-n}\ where Q = \frac{\partial Z}{\partial Y} \rightarrow y^n q = Q(1-n)$$

Now the given equation reduces to $f[(1-m)P,(1-n)Q] = 0$ *which is of the form* $f(P,Q) = 0$

Case(ii):- when $m = 1, n = 1$

Put $X = logx\ and\ Y = logy\ then$

$$p = \frac{\partial z}{\partial x} = \frac{\partial Z}{\partial X}\frac{\partial X}{\partial x} = \frac{\partial Z}{\partial X}\frac{1}{x}\ implies\ \ px = P = where P = \frac{\partial Z}{\partial X}$$

$$similarly\ \ qy = Q \quad where Q = \frac{\partial Z}{\partial Y}$$

the given equation reduces to the form $\boxed{f(P,Q) = 0}$

EQUATIONS OF THE FORM $F(x^m p, y^n q, z) = 0$ (*where m and n are constants*)

This can be reduced to an equation of the form

$f(P,Q,z) = 0$ by the substitutions given for the equation $F(x^m p, y^n q, z) = 0$ as above.

PROBLEMS

1.**Solve the partial differential equation** $\frac{x^2}{p} + \frac{y^2}{q} = z$

Sol. Given equation can be written as

$$x^2 p^{-1} + y^2 q^{-1} = z\ or \left(x^{-2}p\right)^{-1} + \left(y^{-2}q\right)^{-1} = z \rightarrow (1)$$

This is of the form $f\left(x^m p, y^n q, z\right) = 0$ with $m = -2, and\ n = -2.$

$$\text{Put}\ \ X = x^{1-m} = x^{1-(-2)} = x^3 \ \text{and}\ \ Y = y^{1-n} = y^{1+2} = y^3$$

$$\text{Then}\ \ p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P.3x^2 \ \ \text{where}\ \ P = \frac{\partial z}{\partial X}$$

$$\therefore x^{-2}p = 3P$$

$$\text{and}\quad q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y} = Q.3y^2 \ \ \text{where}\ \ Q = \frac{\partial z}{\partial Y}$$

$$\therefore y^{-2}q = 3Q$$

Now equation (1), becomes.

$$(3P)^{-1} + (3Q)^{-1} = z \rightarrow (2)$$

Since (2) is of the form $f(P,Q,z) = 0$

Put $\boldsymbol{P} = \boldsymbol{aQ}\ in\ (1), then\ we\ get\ Q = \frac{(a+1)}{3az}$

$$\therefore P = \frac{(a+1)}{3z}$$

Putting the values of Pand Q in $\boldsymbol{dz} = \boldsymbol{P}\,\boldsymbol{dX} + \boldsymbol{Q}\,\boldsymbol{dY}$,we get

$$\frac{3az}{a+1} dz = (\boldsymbol{adX} + \boldsymbol{dY})$$

Integrating,we get

$$\int \frac{3az}{a+1} dz = (\boldsymbol{a} \int \boldsymbol{dX} + \int \boldsymbol{dY})$$

$$\frac{3az^2}{2(a+1)} = (aX + Y) + c$$

$\therefore 3z^2 = 2\left(\frac{a+1}{a}\right)\left(x^3 + ay^3\right) + c_1$,taking $c_1 = 2(\frac{a+1}{a})c$, is the required solution of (1)

2.**Solve the partial differential equation** $\frac{p}{x^2} + \frac{q}{y^2} = z$

Sol. The given equation can be written as

$$px^{-2} + qy^{-2} = z \rightarrow (1)$$

Since (1) is of the form $f\left(x^m p, y^n q, z\right) = 0$ With $m = -2, and\ n = -2$

Put $X = x^{1-m} = x^3$, and $Y = y^{1-n} = y^3$

$$\text{Now } p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P.3x^2 \text{ where } P = \frac{\partial z}{\partial X}$$

$$\therefore x^{-2} p = 3P$$

$$\text{and } q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y} = Q3y^2 \text{ where } Q = \frac{\partial z}{\partial Y}$$

$$\therefore y^{-2} q = 3Q$$

Equation (1) becomes, $3P + 3Q = z \rightarrow (2)$

Since (2) is of the form $f\left(P, Q, \text{z}\right) = 0$

Put $\boldsymbol{P} = \boldsymbol{aQ}\ in\ (1), then\ we\ get\ Q = \frac{z}{3(a+1)}$

$$\therefore P = \frac{az}{3(a+1)}$$

Putting the values of Pand Q in $\boldsymbol{dz} = \boldsymbol{P}\,\boldsymbol{dX} + \boldsymbol{Q}\,\boldsymbol{dY}$,we get

$$\frac{dz}{z} = \frac{1}{3(a+1)}(adX + dY)$$

Integrating,we get

$$\int \frac{dz}{z} = \frac{1}{3(a+1)}\left(a\int dX + \int dY\right)$$

$$\log z = \frac{1}{3(a+1)}(aX + Y) + C$$

$$\Rightarrow \log z = \frac{1}{3(1+a)}\left(x^3 + ay^3\right) + c$$, is the complete solution of (1)

3.**Solve** $q^2y^2 = z(z - px)$

Sol. Given equation can be written as

$$q^2y^2 = z^2 - zpx \; or \; (xp)z + (qy)^2 = z^2 \rightarrow (1)$$

Since (1) is of the form $f\left(x^m p, y^n q, z\right) = 0$ with $m = 1$ and $n = 1$

Put $X = \log x$ and $Y = \log y$

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P.\frac{1}{x}$ where $P = \frac{\partial z}{\partial X}$

$\therefore xp = P$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y} = Q.\frac{1}{y}$ where $Q = \frac{\partial z}{\partial Y}$

$\therefore qy = Q$

$\therefore$ Equation (1), becomes, $Pz + Q^2 = z^2 \rightarrow (2)$

Since (2) is of the form $f(P, Q, z) = 0$

Put $\boldsymbol{P} = \boldsymbol{aQ}$ in (1), $then$ we get $Q = \frac{z}{2}[-a \pm \sqrt{a^2 + 4}]$

$$\therefore P = \frac{aZ}{2}[-a \pm \sqrt{a^2 + 4}]$$

Putting the values of Pand Q in $\boldsymbol{dz = P\,dX + Q\,dY}$,we get

$$\frac{dz}{z} = \frac{1}{2}[-a \pm \sqrt{a^2 + 4}](adX + dY)$$

Integrating,we get

$$\int \frac{dz}{z} = \frac{1}{2}[-a \pm \sqrt{a^2 + 4}]\left(a\int dX + \int dY\right)$$

$$\log z = \frac{1}{2}\left[-a \pm \sqrt{a^2+4}\right](\boldsymbol{aX}+\boldsymbol{Y})+\boldsymbol{c}$$

$\therefore \log z = \frac{1}{2}\left[-a \pm \sqrt{a^2+4}\right](\boldsymbol{ax^3}+\boldsymbol{y^3})+\boldsymbol{c}$ **,**is the complete integeral of (1)

4.**Solve the partial differential equation** $p^2x^4 + y^2zq = 2z^2$

Sol. Given equation is $p^2x^4 + y^2zq = 2z^2$

Then given equation can be written as

$$\left(px^2\right)^2 + \left(qy^2\right)z = 2z^2 \rightarrow (1)$$

Since (1) is of the form $f\left(x^m p, y^n q, z\right) = 0$ with m=2 and n=2

$$\text{Put } X = x^{1-m} = x^{1-2} = x^{-1} = \frac{1}{x} \text{ and } Y = y^{-1} = \frac{1}{x}$$

$$\text{Now } P = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P.\left(\frac{-1}{x^2}\right), \text{ where } P = \frac{\partial z}{\partial X}$$

$$\therefore x^2 p = -P$$

$$\text{and } q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y} = Q.\left(\frac{-1}{y^2}\right), \text{ where } Q = \frac{\partial z}{\partial Y}$$

$$\therefore y^2 q = -Q$$

Now equation (1) becomes, $P^2 - Qz = 2z^2 \; or \; P^2 - Qz = 2z^2 \rightarrow (2)$

Since (2) is of the form $f(P,Q,z) = 0$

Put $\boldsymbol{P} = \boldsymbol{aQ}$ $in\ (1), then\ we\ get\ Q = \frac{z}{2a^2}\left[1 \pm \sqrt{8a^2+1}\right]$

$$\therefore P = \frac{Z}{2a}\left[1 \pm \sqrt{8a^2+1}\right]$$

Putting the values of Pand Q in $\boldsymbol{dz} = \boldsymbol{P}\,\boldsymbol{dX} + \boldsymbol{Q}\,\boldsymbol{dY}$,we get

$$\frac{dz}{z} = \frac{1}{2a^2}\left[1 \pm \sqrt{8a^2+1}\right](\boldsymbol{adX}+\boldsymbol{dY})$$

Integrating,we get

$$\int \frac{dz}{z} = \frac{1}{2a^2}\left[1 \pm \sqrt{8a^2+1}\right]\left(\boldsymbol{a}\int \boldsymbol{dX} + \int \boldsymbol{dY}\right)$$

$$\log z = \frac{1}{2a^2}\left[1 \pm \sqrt{8a^2+1}\right](\boldsymbol{aX}+\boldsymbol{Y})+\boldsymbol{c}$$

$\therefore \log z = \frac{1}{2a^2}[1 \pm \sqrt{8a^2+1}](\boldsymbol{ax^3} + \boldsymbol{y^3}) + \boldsymbol{c}$, is the complete integral of (1)

5. **Solve** $x^2p^2 + xpq = z^2$

Sol. The given equation can be written as

$$(xp)^2 + (xp)q = z^2 \rightarrow (1)$$

Since (1) is of the form $f\left(x^m p.y^n q, z\right) = 0$ with m=1 and n=0

Put $X = \log x$

$$\text{Now } P = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P\frac{1}{x}, \text{ where}$$

$$P = \frac{\partial z}{\partial X}$$

$$\therefore xp = P$$

Equation (1) becomes, $P^2 + Pq = z^2 \rightarrow$ (2)

Since (2) is of the form $f(P, q, z) = 0$

Put $P = aq \text{ in } (2), we \text{ get}$

$$q = \frac{z}{\sqrt{a(a+1)}}, \quad P=a\frac{z}{\sqrt{a(a+1)}}$$

But we have

$$\boldsymbol{dz = P\, dX + q\, dy}$$

Substuting P,q ,we get

$$\frac{\boldsymbol{dz}}{\boldsymbol{z}} = \frac{1}{\sqrt{a(a+1)}}(\boldsymbol{a\, dX + dy})$$

Integrating on both sides

$$\int dz/z = \frac{1}{\sqrt{a(a+1)}}(\boldsymbol{a} \int \boldsymbol{dX + d\, y})$$

$\therefore \sqrt{a(a+1)} \log z = (aX + y) + C$,is the complete integral of (1) .

6.Solve $z = p^2x + q^2y$

Sol. Given equation is $z = p^2x + q^2y$

The given equation can be written as

$$\left(p\sqrt{x}\right)^2 + \left(q\sqrt{y}\right)^2 = z \; or \left(px^{\frac{1}{2}}\right)^2 + \left(qy^{\frac{1}{2}}\right)^2 = z \rightarrow (1)$$

This is of the form $f\left(x^m p, y^n q, z\right) = 0 \; with \; m = n = \frac{1}{2}$

$$\text{Put } X = x^{1-m} = x^{1-\frac{1}{2}} = x^{\frac{1}{2}} \; and \; Y = y^{1-\frac{1}{2}} = y^{\frac{1}{2}}$$

$$\text{Now } p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P\left(\frac{1}{2}x^{\frac{-1}{2}}\right), where \; P = \frac{\partial z}{\partial X}$$

$$\text{and } q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial x} = Q\left(\frac{1}{2}y^{\frac{-1}{2}}\right), where \; Q = \frac{\partial z}{\partial Y}$$

$$\therefore px^{\frac{1}{2}} = \frac{P}{2} \; and \; qy^{\frac{1}{2}} = \frac{Q}{2}$$

Then equation (1) becomes, $\left(\frac{P}{2}\right)^2 + \left(\frac{Q}{2}\right)^2 = z \; i.e. P^2 + Q^2 = 4z \rightarrow (2)$

This if of the form $f\left(P, Q, z\right) = 0$

Put $P = aQ \; in \; (2), we \; get$

$$a^2Q^2 + Q^2 = 4z$$

$$Q = \sqrt{\frac{4z}{a^2+1}}, \quad P = a\sqrt{\frac{4z}{a^2+1}},$$

$$\boldsymbol{dz = P\,dX + Q\,dY}$$

Substuting P,Q ,we get

$$\boldsymbol{dz} = \sqrt{\frac{4z}{a^2+1}}(\boldsymbol{a\,dX + dY})$$

$$\frac{dz}{\sqrt{z}} = \frac{2}{\sqrt{a^2+1}}(\boldsymbol{a\,dX + dY})$$

Integrating on both sides , $\int dz/\sqrt{z} = \frac{2}{\sqrt{a^2+1}}(\boldsymbol{a \int dX + \int dY})$

$$\sqrt{(a^2+1)}\sqrt{z} = (aX + Y) + C$$

$\sqrt{(a^2+1)}\sqrt{z} = \left(a\sqrt{x}+\sqrt{y}\right) + C$, is the complete integral of (1)

7.Solve $x^2p^2 + y^2q^2 = z^2$

Sol. Given *equation is* $x^2p^2 + y^2q^2 = z^2 \ldots\ldots(1)$

$$(xp)^2 + (yq)^2 = z^2$$

Equation (1), becomes , $P^2 + Q^2 = z^2 \ldots\ldots(2)$

$$Put\ P = aQ\ in\ (2), we\ get\ Q = \frac{z}{\sqrt{a^2+1}}; P = \frac{az}{\sqrt{a^2+1}}$$

But we have , $dz = P\,dX + Q\,dY$

Substuting P,Q ,we get

$$dz = \frac{z}{\sqrt{a^2+1}}(a\,dX + dY)$$

$$\frac{dz}{z} = \frac{1}{\sqrt{a^2+1}}(a\,dX + dY)$$

Integrating on both sides

$$\int dz/z = \frac{1}{\sqrt{a^2+1}}\left(a\int dX + \int dY\right)$$

$$\sqrt{(a^2+1)}\log z = (aX+Y) + C$$

$\sqrt{(a^2+1)}\log z = (a\log x + \log y) + C$,is the Complete solution of (1)

8.Solve $x^2p^2 + y^2q^2 = 1$

Sol. Given equation is $x^2p^2 + y^2q^2 = 1 \ldots\ldots(1)$

$$(xp)^2 + (yq)^2 = 1$$

Since (1) is of the form $f(x^m p, y^n q) = 0$ with $m = 1\ and\ n = 1$

Put $X = \log x$ and $Y = \log y$

$$\text{Now } p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P.\frac{1}{x} \text{ where } P = \frac{\partial z}{\partial X}$$

$$\therefore xp = P$$

$$\text{and } q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y} = Q.\frac{1}{y} \text{ where } Q = \frac{\partial z}{\partial Y}$$

$\therefore qy = Q$

$\therefore$ Equation (1), becomes

$$P^2 + Q^2 = 1 \quad \text{......(2)}$$

$$Put\ P = a\ in\ (2), we\ get\ Q = \sqrt{1-a^2}$$

But we have

$$\boldsymbol{dz = P\,dX + Q\,dY}$$

Substuting P,Q ,we get

$$\boldsymbol{dz = (a\,dX + \sqrt{1-a^2}\,dY)}$$

Integrating on both sides

$$\int \boldsymbol{dz} = (\boldsymbol{a} \int \boldsymbol{dX} + \sqrt{1-a^2} \int \boldsymbol{dY})$$

$$z = \left(aX + \sqrt{1-a^2}Y\right) + C$$

z=$\left(a \log x + \sqrt{1-a^2} \log y\right) + C$,is the Complete solution of (1)

EQUATIONS OF THE FORM $\boldsymbol{F(z^n p, z^n q) = 0 \ \ (where\ \ n\ is\ a\ \ constant)}$

Use the following substitution to reduce the above form to an equation of the form f(P,Q)=0

$$put \quad Z = \begin{cases} z^{n+1} & if\ n \neq -1 \\ \log z, & if\ n = -1 \end{cases}$$

EQUATIONS OF THE FORM $\boldsymbol{f(x, z^n p) = g(y, z^n q) \ \ (where\ \ n\ is\ a\ \ constant)}$

An equation of the above form can be reduced to an equation of the form f(P,Q)=0

by the substitutions given for the equation $F(z^n p, z^n q) = 0$ as above .

PROBLEMS

1. **Solve** $z^2\left(p^2+q^2\right)=x^2+y^2$

Sol. Given equation is $z^2\left(p^2+q^2\right)=x^2+y^2$

The given equation can be written as

$$z^2p^2+z^2q^2=x^2+y^2 \text{ or } z^2p^2-x^2=y^2-z^2q^2$$

$$\text{Or } (zp)^2-x^2=y^2-(zq)^2 \rightarrow (1)$$

Since (1) is the of the form $f\left(x, pz^n\right)=g\left(y, qz^n\right)$. with n=1

$$\therefore put\, Z=z^{n+1}=z^{1+1}=z^2$$

$$\text{Then } \frac{\partial Z}{\partial x}=2z.\frac{\partial z}{\partial x} \Rightarrow P=2zp \text{ where } P=\frac{\partial z}{\partial x}$$

$$\therefore pz=\frac{P}{2}$$

$$\text{and } \frac{\partial Z}{\partial y}=2z.\frac{\partial z}{\partial y} \Rightarrow Q=2zq \text{ where } Q=\frac{\partial z}{\partial y} \therefore qz=\frac{Q}{2}$$

$\therefore$ Equation (1) becomes, $\frac{P^2}{4}-x^2=y^2-\frac{Q^2}{4}$

$$i.e., P^2-4x^2=4y^2-Q^2 \rightarrow (2)$$

This is of the form $f_1(x,P)=f_2(y,Q)$

$$\text{Let } P^2-4x^2=4y^2-Q^2=4k^2 \text{ (say)}$$

$$\therefore P^2-4x^2=4k^2 \text{ and } 4y^2-Q^2=4k^2$$

$$\Rightarrow P^2=4x^2+4k^2 \text{ and } Q^2=4y^2-4k^2$$

$$\therefore P=2\sqrt{x^2+k^2} \text{ and } Q=2\sqrt{y^2-k^2}$$

$$\text{We have } dZ=\frac{\partial Z}{\partial x}.dx+\frac{\partial Z}{\partial y}dy$$

$$=Pdx+Qdy \text{ [By total differentiation]}$$

$$\therefore dZ=2\sqrt{x^2+k^2}dx+2\sqrt{y^2-k^2}dy$$

Integrating on both sides

$$Z=2\int\sqrt{x^2+k^2}dx+2\int\sqrt{y^2-k^2}dy$$

$$=2\left[\frac{x}{2}\sqrt{x^2+k^2}+\frac{k^2}{2}\sinh^{-1}\left(\frac{x}{k}\right)\right]+2\left[\frac{x}{2}\sqrt{y^2-k^2}-\frac{k^2}{2}\cosh^{-1}\left(\frac{y}{k}\right)\right]+c$$

$$=x\sqrt{x^2+k^2}+k^2\sinh^{-1}\left(\frac{x}{k}\right)+x\sqrt{y^2-k^2}+k^2\cosh^{-1}\left(\frac{y}{k}\right)+c$$

$$or\ z^2=x\sqrt{x^2+k^2}+y\sqrt{y^2-k^2}+k^2\left[\sinh^{-1}\left(\frac{x}{k}\right)-\cosh^{-1}\left(\frac{y}{k}\right)\right]+c$$

$or\ z^2=x\sqrt{x^2+k^2}+y\sqrt{y^2-k^2}+k^2\log\left(\frac{x+\sqrt{x^2+k^2}}{y+\sqrt{y^2-k^2}}\right)+c$ is the complete solution of (1)

2. **Solve the partial differential equation.** $p^2z^2\sin^2 x+q^2z^2\cos^2 y=1$

Sol. Given equation is $p^2z^2\sin^2 x+q^2z^2\cos^2 y=1$

The given equation can be written as

$$(pz)^2\sin^2 x+(qz)^2\cos^2 y=1\ or\ (pz)^2\sin^2 x=1-(qz)^2\cos^2 y\rightarrow(1)$$

Since (1) is of the form $f\left(x,pz^n\right)=g\left(y,qz^n\right)$ with n=1.

Put $Z=z^{n+1}=z^2$

Now $\frac{\partial Z}{\partial x}=2z.\frac{\partial z}{\partial x}\Rightarrow P=2zp\ or\ pz=\frac{P}{2}$ where $P=\frac{\partial z}{\partial x}; Q=\frac{\partial z}{\partial y}$

and $\frac{\partial Z}{\partial y}=2z.\frac{\partial z}{\partial y}\Rightarrow Q=2zq\ or\ qz=\frac{Q}{2}$

Then equation (1) becomes, $\left(\frac{P}{2}\right)^2\sin^2 x=1-\left(\frac{Q}{2}\right)^2\cos^2 y$

$i.e.\frac{P^2}{4}\sin^2 x=1-\frac{Q^2}{4}\cos^2 y\rightarrow(2)$

This is of the form $f_1(x,p)=f_2(y,q)$

Let $\frac{P^2}{4}\sin^2 x=1-\frac{Q^2}{4}\cos^2 y=k^2$ (constant)

$$\therefore\frac{P^2}{4}\sin^2 x=k^2\ and\ 1-\frac{Q^2}{4}\cos^2 y=k^2$$

$$\Rightarrow P^2\sin^2 x=4k^2\ and\ Q^2\cos^2 y=4\left(1-k^2\right)$$

$$\Rightarrow P = \frac{2k}{\sin x} \, and \, Q = \frac{2\sqrt{1-k^2}}{\cos y}$$

We have $dZ = \frac{\partial Z}{\partial x}dx + \frac{\partial Z}{\partial y}dy$ [By total differential]

$$\therefore dZ = Pdx + Qdy$$

$$dZ = \frac{2k}{\sin x}dx + \frac{2\sqrt{1-k^2}}{\cos y}dy$$

Integrating on both sides

$$z = 2k \int \csc x \, dx + 2\sqrt{1-k^2} \int \sec y \, dy$$

$$= 2k \log\left(\operatorname{cosec} x - \cot x\right) + 2\sqrt{1-k^2} \log\left(\sec y + \tan y\right) + c$$

$\therefore z^2 = 2k \log\left(\operatorname{cosec} x - \cot x\right) + 2\sqrt{1-k^2} \log\left(\sec y + \tan y\right) + c$, is the required complete solution of (1)

3.Solve $(x + pz)^2 + (y + qz)^2 = 1$

Sol.Given *equation is* $(x + pz)^2 + (y + qz)^2 = 1$.......(1)

since (1) *is of the form* $F(z^n p, z^n q, x, y) = 0 \quad n = 1$

Put $Z = z^{n+1} = z^2$

Differentiating partially w.r.t 'x',we get $\frac{\partial Z}{\partial z} = 2z$ *implies that* $\frac{\partial z}{\partial Z} = \frac{1}{2z}$

But $p = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial x} = \frac{P}{2z}$ implies $\frac{\partial z}{\partial x} = \frac{P}{2} = zp$; *Similarly we get* $qz = \frac{Q}{2}$

Substitute in (1),we get

$$(x + \tfrac{P}{2})^2 + (y + \tfrac{Q}{2})^2 = 1$$

Separating P *and* x *from* Q *and* y, the given equation can be written as.

$$(x + \frac{P}{2})^2 = 1 - (y + \frac{Q}{2})^2 = K^2$$

$$(x + \frac{P}{2})^2 = K^2 \; AND \; 1 - (y + \frac{Q}{2})^2 = K^2$$

$$\left(x + \frac{P}{2}\right) = K$$

$$Q = 2(\sqrt{(1-K^2)} - y)$$

$$P = 2(K - x)$$

$$dZ = \frac{\partial Z}{\partial x}dx + \frac{\partial Z}{\partial y}dy$$

$$\therefore dz = 2(k-x)dx + 2\left[\sqrt{1-k^2} - y\right]dy$$

Integrating on both sides

$$z = 2\int (k-x)dx + 2\int\left[\sqrt{1-k^2} - y\right]dy + c$$

$$z = 2(kx - \frac{x^2}{2}) + 2\left[\left(\sqrt{1-k^2}\right)y - \frac{y^2}{2}\right] + c$$

$\therefore z = 2kx - x^2 + 2\left(\sqrt{1-k^2}\right)y - y^2 + c$, is the complete solution of (1)

4..**Solve $z(p^2 - q^2) = x - y$**

Sol. Given equation is $\quad z(p^2 - q^2) = x - y \dots\dots\dots\dots (1)$

$$(z^{\frac{1}{2}}p)^2 - (z^{\frac{1}{2}}q)^2 = x - y \text{.....(2)}$$

$$since\ (2)\ is\ of\ the\ form\ F(z^n p, z^n q, x, y) = 0 \quad n = \frac{1}{2}$$

Put $Z = z^{n+1} = z^{\frac{3}{2}}$

Differentiating partially w.r.t ‘x’,we get $\frac{\partial Z}{\partial z} = \frac{3}{2}z^{\frac{1}{2}}$

$$implies\ that\ \frac{\partial z}{\partial Z} = \frac{2}{3z^{\frac{1}{2}}}$$

But $p = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial x}$=Pimplies $\frac{2}{3}P = z^{\frac{1}{2}}p; Similarly\ we\ get\ \frac{2}{3}Q = z^{\frac{1}{2}}q$

Substitute in (2),we get

$$(\frac{2}{3}P)^2 - (\frac{2}{3}Q)^2 = x - y$$

Separating $P\ and\ x\ from\ Q\ and\ y,$

the given equation can be written as.

$$(\frac{2}{3}P)^2 - x = -y + (\frac{2}{3}Q)^2 = k$$

Solving, we get

$$P = \frac{3}{2}\sqrt{k+x} \text{ and } Q = \frac{3}{2}\sqrt{k+y}$$

We have $dZ = \frac{\partial Z}{\partial x}dx + \frac{\partial Z}{\partial y}dy$ [By total differential]

$$\therefore dZ = Pdx + Qdy$$

$$dZ = \frac{3}{2}[\sqrt{k+x}dx + \sqrt{k+y}dy]$$

Integrating on both sides

$$Z = \frac{3}{2}[\int \sqrt{k+x}dx + \int \sqrt{k+y}\ dy$$

$z^{\frac{3}{2}} = (k+x)^{\frac{3}{2}} + (k+y)^{\frac{3}{2}} + c$, is the required complete solution of (1) .

METHOD OF SEPARATION OF VARIABLES

This method is used to reduce one partial differential equation to two or more ordinary differential equations,each one involving one of the independent variables.This will be done by separating these variables from the beginning. This method is explained through following examples.

1. **Solve by the method of separation of variables $\frac{\partial U}{\partial x} = 2\frac{\partial U}{\partial t} + U$ where U(x,0)=6e^{-3x}**

Sol.Given equation is $\frac{\partial U}{\partial x} = 2\frac{\partial U}{\partial t} + U$----------(1)

Let U(x,t)= X(x) T(t) =XT ---------------(2) , be a solution of (1)

Differentiating (2) partially w.r.t x ant t

$$\frac{\partial U}{\partial x} = X'T \quad , \quad \frac{\partial U}{\partial t} = T'X$$

Put these values in equation (1), we have

$$X'T = 2\,T'X + XT \quad \text{Dividing by XT}$$

$$\frac{X'}{X} = 2\frac{T'}{T} + 1 \text{--------(3)}$$

Since L.H.S is a function of 'x' and the R.H.S is a function of 't' where x and t are independent variables, the two sides of (3) can be equal to each other for all values of 'x' and 't' if and only if both sides are equal to a constant.

Therefore $\frac{X'}{X} = 2\frac{T'}{T} + 1$=k--------(4) where k is a constant

Now from (4) $\frac{X'}{X}$ = k----(5) and $2\frac{T'}{T} + 1$=k-------(6)

Now consider (5) $\frac{X'}{X}$ = k $\Rightarrow$ $X' - kX = 0$ $\Rightarrow X = C_1 e^{kx}$

Now consider (6) $2\frac{T'}{T} + 1$=k $\Rightarrow$ $T' - \left(\frac{k-1}{2}\right)T = 0$ $\Rightarrow T = C_2 e^{\left(\frac{k-1}{2}\right)t}$ ---(8)

Substituting the values of X and T in (2) we get

$$U(x,t) = X = C_1 e^{kx} C_2 e^{\left(\frac{k-1}{2}\right)t}$$

$$U(x,t) = X = Ae^{kx} e^{\left(\frac{k-1}{2}\right)t} \quad (where\ A=C_1 C_2)$$

Put t=0 in the above equation ,we have U(x,0) = A e^{kx} -----(9)

but given that U(x,0)=6e^{-3x}--------(10)

from (9) and (10) we have A e^{kx} =6e^{-3x}

∴ A=6 and k=-3 ,hence the solution of the given equation is

$$U(x,t) = X = 6e^{-3x} e^{(-2)t} = 6e^{-(3x+2t)}$$

2. Solve the equation by the method of separation of variables $\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial y} + 2U$

Sol. Given equation is $\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial y} + 2U$ ----------(1)

Let U(x,y)= X(x) Y(y) =X Y ---------------(2) , be a solution of (1)

Differentiating (2) partially w.r.t x ant y

$$\frac{\partial U}{\partial x} = X'Y \quad , \quad \frac{\partial U}{\partial y} = Y'X \quad \frac{\partial^2 U}{\partial x^2} = X''Y$$

Put these values in equation (1), we have

$$X''Y = Y'X + 2XY$$

Dividing by XY on both sides we have $\frac{X''}{X} = \frac{Y''}{Y}$+2

$$\frac{X''}{X} - 2 = \frac{Y''}{Y} \text{-----------------(3)}$$

Since L.H.S is a function of 'x' and the R.H.S is a function of 'y' where x and y are independent variables, the two sides of (3) can be equal to each other for all values of 'x' and 'y' if and only if both sides are equal to a constant.

$$\frac{X''}{X} - 2 = \frac{Y''}{Y} = k \text{--------(4)}$$

Now from (4)

$$\frac{X''}{X} - 2 = k \text{ -------(5)}$$

And

$$\frac{Y''}{Y} = k \text{ -------(6)}$$

From (5) $X'' - 2X = kX$ $\quad X'' - (2+k)X = 0$

Which is second order differential equation

Auxiliary equation is $m^2 - (2+k) = 0 \quad \rightarrow \quad m = \pm\sqrt{(2+k)}$

Solution of the given equation (5) is $X = C_1 e^{\sqrt{(2+k)}}$

Now consider equation (6) $Y' = kY \quad \rightarrow \frac{Y'}{Y} = k$

Integrating on both sides we get $logy = ky + logC_3$

$$\Rightarrow \log\left(\frac{Y}{C_3}\right) = ky \quad \Rightarrow Y = C_3 e^{ky} \text{ ---(8)}$$

Substituting the values of X and Y in (2) we have

$$U = \left[C_1 e^{\sqrt{(2+k)}x} + C_2 e^{-\sqrt{(2+k)}x}\right] C_3 e^{ky}$$
$$U = \left[A e^{\sqrt{(2+k)}x} + B e^{-\sqrt{(2+k)}x}\right] e^{ky}$$

Where $A = C_1 C_3 \quad and \quad B = C_1 C_2$

APPLICATIONS OF PDE

ONE DIMENSIONAL WAVE EQUATION

Let OA be a stretched string of length l with fixed ends O and A. Let us take x-axis along OA and y-axis along OB perpendicular to OA, with O as origin. Let us assume that the tension T in the string is constant and large when compared with the weight of the string so that the effects of gravity are negligible. Let us pluck the string in the BOA plane and allow it to vibrate. Let p be any point of the string at time t. Let there be no external forces acting on the string. Let each point of the string make small vibrations at right angles to OA in the plane of BOA. Draw pp^1 perpendicular to OA. Let $op^1 = x$ and $pp^1 = y$. Then y is a function of x and t. Under the assumptions, using Newton's second law of motion, it can be proved that $y(x,t)$ is governed by the equation,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} -----------(1)$$

$$i,e., \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

$$where \;\; c^2 = T/m$$

With T = tension in the string at any point and m is mass per unit length of the string.

Since the points O and A are not disturbed from their original positions for any time t we get $y(0,t) = 0------(2)$ $\quad y(1,t) = 0------(3)$

These are referred to as the end conditions or boundary conditions. Further it is possible that, we describe the initial position of the string as well as the initial velocity at any point of the string at time $t = 0$ through the conditions

$$y(x,0) = f(x), 0 \le x \le l ------(4)$$

$$\frac{\partial y}{\partial t}(x,0) = g(x), 0 \le x \le l -----(5)$$

Where $f(x)$ and $g(x)$ are functions such that $f(0) = f(l) = 0$;and $g(0) = g(l) = 0$. Thus to study the subsequent motion of any point of $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} -----(1)$ the string we have to solve following :Determine $y(x,t)$ such that Subject to the condition

$$\left.\begin{aligned} y(0,t)&=0 \ \ for\ all\ t----(2)\\ y(l,t)&=0 \ \ for\ all\ t----(3)\end{aligned}\right\}end\ conditions$$

$$\left.\begin{aligned} y(x,0) &= f(x), 0\le x\le 1----(4)\\ \left(\frac{\partial y}{\partial t}\right)_{at\ t=0} &= g(x), 0\le x\le 1----(5)\end{aligned}\right\}initial\ conditions$$

The equation (1) is called one dimensional wave equation

Solution of equation (1) subject to the conditions equation (2) to (5)

Consider the equation $\frac{\partial^2 y}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}------(1)$

Let us use the method of separation of variables. Here $y=y(x,t)$. Let us take $y=X(x)T(t)$

as solution of (1). Then

$$\frac{\partial y}{\partial x}=X^1(x)T(t);\frac{\partial^2 y}{\partial x^2}=X^{11}(x)T(t);$$
$$\frac{\partial y}{\partial t}=X(x)T^1(t);\frac{\partial^2 y}{\partial t^2}=X(x)T^{11}(t)$$

Using these in (1) we get

$$X^{11}(x)T(t)=\frac{1}{c^2}X(x)T^{11}(t)$$
$$\therefore\frac{X^{11}(x)}{X(x)}=\frac{1}{c^2}\frac{T^{11}(t)}{T(t)}$$

Since the left hand side is a function of x and right hand side is a function of t the equality is possible if and only if each side is equal to the same constant (say) λ.

Hence we shall take $\frac{X^{11}(x)}{X(x)}=\frac{1}{c^2}\frac{T^{11}(t)}{T(t)}=\lambda$

Let us take λ to be real. Then three cases are possible $\lambda>0, \lambda=0\ or\ \lambda<0$

<u>Case 1:-</u> Let $\lambda>0$, then $\lambda=p^2(p>0)$

Then $\frac{X^{11}(x)}{X(x)}=\frac{1}{c^2}\frac{T^{11}(t)}{T(t)}=p^2$

Hence $X^{11}(x)=p^2X(x)(i.e.,)X^{11}(x)-p^2X(x)=0$

$$i.e., \frac{d^2X}{dx^2} - p^2X = 0 \Rightarrow X(x) = A_1e^{px} + B_1e^{-px}$$
$$Also\ T^{11}(t) - p^2c^2T(t) = 0$$
$$\Rightarrow T(t) = C_1e^{pct} + D_1e^{-pct}$$

Hence in this case, a typical solution is like

$$y(x,t) = \left(A_1e^{px} + B_1e^{-px}\right)\left(C_1e^{pct} + D_1e^{-pct}\right) ------ (S.1)$$

Where A_1, B_1, C_1, D_1 are arbitary constants .

Case 2:- let $\lambda = 0$ then

$$\frac{X^{11}(x)}{X(x)} = \frac{T^{11}(t)}{c^2T(t)} = 0 \quad \therefore X^{11}(x) = 0 \Rightarrow X(x) = A_2 + B_2x$$
$$T^{11}(t) = 0 \Rightarrow T(t) = C_2 + D_2t$$
$$\therefore y(x,t) = (A_2 + B_2x)(C_2 + D_2t) + ------ (S.2)$$

Where A_2, B_2, C_2, D_2 are arbitary constants.

Case 3:- Let $\lambda < 0$. Then we can write $\lambda = -p^2$ where $p > 0$ then

$$\frac{X^{11}(x)}{X(x)} = \frac{T^{11}(t)}{c^2T(t)} = -p^2$$
$$\therefore X^{11}(x) + p^2X(x) = 0$$
$$\Rightarrow X(x) = (A_3 \cos px + B_3 \sin px)$$
$$T^{11}(t) + p^2c^2T(t) = 0$$

$$\Rightarrow X(t) = (C_3 \cos pct + D_3 \sin pct)$$

Hence a typical solution in this case is

$$y(x,t) = (A_3 \cos px + B_3 \sin px)(C_3 \cos pct + D_3 \sin pct)$$

Thus the possible solution forms of equation (1) are

$$y(x,t) = \left(A_1e^{px} + B_1e^{-px}\right)\left(C_1e^{pct} + D_1e^{-pct}\right) ---- (S.1)$$
$$y(x,t) = (A_2 + B_2x)(C_2 + D_2t) ------- (S.2)$$
$$y(x,t) = (A_3 \cos px + B_3 \sin px)(C_3 \cos pct + D_3 \sin pct) --- (S.3)$$

Consider $(S.1)$ $y(x,t)=\left(Ae^{px}+Be^{-px}\right)\left(Ce^{pct}+De^{-pct}\right)$

Using conditions (2) $(viz)\, y(0,t)=0 \text{ for all } t$

$$(A+B)\left(Ce^{pct}+De^{-pct}\right)=0 \text{ for all } t \therefore A+B=0$$

Using condition (3), $y(l,t)=0 \text{ for all } t$

$$\therefore \left(Ae^{pl}+Be^{-pl}\right)\left(Ce^{pct}+De^{pct}\right)=0 \text{ for all } t$$
$$\therefore Ae^{pl}+Be^{-pl}=0$$

Solving $A+B=0$ And $Ae^{pl}+Be^{-pl}=0$ We get $A=B=0$ Thus

$$y(x,t)=0$$

This implies that there is no displacement for any x and for any t. This is impossible. Thus $(S.1)$ is not an appropriate solution .

Consider $(S.2)$:

$$y(x,t)=(A+Bx)(C+Dt)$$

Using (2), $y(0,t)=0 \text{ for all } t$

Hence $A(C+Dt)=0 \Rightarrow A=0$

Using (3), $y(l,t)=0 \text{ for all } t$

$$\therefore (A+Bl)(C+Dt)=0 \text{ for all } t$$

$\therefore Bl(C+Dt)=0 \forall t$ since A = 0

Here $l\neq 0; C+Dt\neq 0 \forall t$ Hence B = 0

Thus here again $y(x,t)\equiv 0 \forall x \text{ and } t$; Thus as before, this solution also is not valid

Hence $(S.2)$ is also not appropriate for the present problem

Consider $(S.3)$

$$y(x,t) = (A\cos px + B\sin px)(C\cos pct + D\sin pct) \text{ (using condition 2)}$$

$$y(x,t) = 0 \forall t$$
$$\Rightarrow A(C\cos pct + D\sin pct) = 0 \forall t$$
$$\Rightarrow A = 0$$

Using condition (3)

$$y(l,t) = 0 \forall t$$

$$B\sin pl(C\cos pct + D\sin pct) = 0$$

If $B = 0, y(x,t) = 0$ and this is invalid

Hence $\sin pl = 0$

$\therefore pl = n\pi$ Where n = 1, 2, 3…………

Thus $p = \frac{n\pi}{l}(n = 1, 2, 3, ...)$

Thus a typical solution of (1) satisfying conditions (2) & (3) is

$$y(x,t) = \sin\frac{n\pi x}{l}\left[C_n \cos\frac{n\pi ct}{l} + D_n \sin\frac{n\pi ct}{l}\right]$$
for $n = 1, 2, 3......$

Since different solutions correspond to different positive integer n.

AN IMPORTANT OBSERVATION HERE :

If $[y_n(x,t)]_{n=1}^{\infty}$ are functions satisfying (1) as well as conditions (2) and (3), as the equation (1) is linear, the most general solution of (1) here is $y(x,t) = \sum_{n=1}^{\infty} y_n(x,t)$

Thus the most general solution of (1) satisfying (2) & (3) is

$$y(x,t) = \sum_{n=1}^{\infty}\left[C_n \cos\frac{n\pi ct}{l} + D_n \sin\frac{n\pi ct}{l}\right]\sin\frac{n\pi x}{l} \rightarrow (6)$$

Where C_n *and* D_n are arbitrary constants to be determined

Let us use condition 4: $y(x,0)=f(x), 0\le x\le l$ Thus putting t = 0 in (6)

$$\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} = f(x), 0\le x\le l$$

Hence $C_n = 2/l\int_0^l f(x)\sin\frac{n\pi x}{l}dx$ n = 1, 2,…. ……………….(7)

Thus $C_n's$ are all determined

Let us consider condition (5):

$$\left(\frac{\partial y}{\partial t}\right)_{at\, t=0} = g(x)\forall 0\le x\le l$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\left\{\left(-C_n \sin\frac{n\pi ct}{l}\left(\frac{n\pi c}{l}\right) + D_n \cos\frac{n\pi ct}{l}\left(\frac{n\pi c}{l}\right)\right)\sin\frac{n\pi x}{l}\right\}$$

$$\left[\frac{\partial y}{\partial t}\right]_{at\, t=0} = g(x)$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(D_n\frac{n\pi c}{l}\right)\sin\frac{n\pi x}{l} = g(x), 0\le x\le l$$

Hence $D_n = \frac{2}{n\pi c}\int_0^t g(x)\sin\frac{n\pi x}{l}dx \;\; for(n=1,2.....)$.................(8) Thus D_n are all determined

Hence the displacement $y(x,t)$ at any point x and at any subsequent time t is given by

$$y(x,t) = \sum\left(C_n\cos\frac{n\pi ct}{l} + D_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l} \rightarrow (6)$$

Where $C_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx \rightarrow (7)$ $\quad D_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}dx \rightarrow (8)$

TWO DIMENSIONAL WAVE EQUATION

Two dimensional wave equation is given by $\frac{\partial^2 u}{\partial t^2} = C^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$-----(1)

Where $C^2 = T/P$, for the unknown displacement $u(x,y,t)$ of a point (x,y) of the vibrating membrane from rest $(\mu = 0)$ at time t.

The boundary conditions (membrane fixed along the boundary in the xy- plane) for all times $t\ge 0$, are $u=0$ on the boundary ----(2)

And the initial conditions are $u(x,y,0)=f(x,y):u_t(x,y,o)=g(x,y)----(3)$

$where\ u_t=\frac{\partial u}{\partial t}$

Now we have to find a solution of the partial differential equation (1) satisfying the conditions (2) and (3) . We shall do this in 3 steps, as follows:

Working Rule To Solve Two – Dimensional Wave Equation

Step1: By the "method of separating variables" setting $u(x,y,t)=F(x,y),G(t)$ and later $F(x,y)=H(x)Q(y)$,we obtain from (1) an ordinary differential equation for G and one partial differential equation for F, two ordinary differential equations for H & Q.

Step 2: We determine solutions of these equations that satisfy the boundary conditions (2). Step(2) to obtain a solution of (1) satisfying both (2) and (3). That is the solution of the rectangular membrane as follows.

Step 3: Finally, using double fourier series, we compose the solutions obtained in step(2).

The double Fourier series for $f(x,y)=[u(x,y,0)]$ is given by

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}u_{mn}(x,y,t)$$

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[B_{mn}\cos\lambda_{mn}t+B^{*}{}_{mn}\sin\lambda_{mn}t\right]\sin\frac{n\pi x}{a}\sin\frac{n\pi y}{b}$$

Hence B_{mn} and $B^{*}{}_{mn}$ are called Fourier co-efficient of $f(x,y)$ and are given by

$$B_{mn}=\frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\frac{n\pi x}{a}\sin\frac{n\pi y}{b}dxdy, m=1,2.......;n=1,2.......$$

$$and\ B^{*}{}_{mn}=\frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a g(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy, m=1,2.......;n=1,2.......$$

PROBLEMS

1. **Find the solution of the wave equation** $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ **corresponding to the triangular initial deflection**

$$f(x) = \frac{2kx}{l} \; where \; 0 < x < l/2$$
$$= \frac{2k}{l}(l-x) \; where \; l/2 < x < l$$

and initial velocity equal to 0.

Sol To find $u(x,t)$ we have to solve $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \rightarrow (1)$

$$u(0,t) = 0 \forall t \rightarrow (2)$$

Where $u(l,t) = 0 \forall t \rightarrow (3)$

$$u(x,0) = f(x)(0 \le x \le l) \rightarrow (4)$$

$$\left(\frac{\partial u}{\partial t}\right)_{at\; t=0} = g(x) = 0(0 \le x \le l) \rightarrow (5)$$

Equation (1) can be in the form

$u(x,t) = T(t)X(x)$ The three solutions of (1) are

$$u(x,t) = \left(A_1 e^{px} + B_1 e^{-px}\right)\left(C_1 e^{pct} + D_1 e^{-pct}\right) ----(S.1)$$
$$u(x,t) = (A_2 + B_2 x)(C_2 + D_2 t) ----(S.2)$$
$$u(x,t) = (A_3 \cos px + B_3 \sin px)(C_3 \cos pct + D_1 \sin pct) ----(S.3)$$

The appropriate solution is S.3

Hence $u(x,t) = (A\cos px + B\sin px)(C\cos pct + D\sin pct)$

Using (2) & (3)

$$A = 0; P = \frac{n\pi}{l} \; where \; n = 1,2,3......$$

$\therefore$ The most general solution of (1) satisfying (2) & (3) is

$$u(x,t) = \sum_{n=1}^{\infty}\left(C_n \cos\frac{n\pi ct}{l} + D_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l} \rightarrow (6)$$

Using (4)

$$u(x,0) = f(x)$$

$$\therefore f(x) = \sum C_n \sin\frac{n\pi x}{l} \; \forall x \in [0,l] \rightarrow (7)$$

Now we can expand the given function $f(x)$ in a half range Fourier sine series for $0 < x < l$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad where \; b_n = \frac{2}{l}\int_0^l f(x).\sin\frac{n\pi x}{l}dx \rightarrow (8)$$

Comparing (7) & (8) we get $c_n = b_n$

$$\therefore c_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} \frac{2k}{l} x\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l} \frac{2k}{l}(l-x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{4k}{l^2}\left[\left\{x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}}\right)\right\}_0^{l/2} + \left\{(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}}\right)\right\}_{l/2}^{l}\right]$$

$$= \frac{4k}{l^2}\left[l/2.\frac{1}{n\pi} - \cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} - \left\{\frac{1}{2}.\frac{1}{n\pi}(-\cos\frac{n\pi}{2}) - \frac{l^2}{n^2\pi^2}.\sin\frac{n\pi}{2}\right\}\right]$$

$$= \frac{4k}{l^2}.2.\frac{l^2}{n^2\pi^2}.\sin\frac{n\pi}{2}$$

$$= \frac{8k}{n^2\pi^2}.\sin\frac{n\pi}{2}$$

If $n = 2m$ (an even number) $C_{2m} = 0$

If $n = 2m+1(an\ odd\ number), C_{2m+1} = \dfrac{8k}{(2m+1)^2\pi^2}(-1)^m$

Thus all C_n's are determined

Using

$$\left[\frac{\partial u}{\partial t}\right]_{t=0} = g(x) \; for \; 0 \le x \le l$$

$$D_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}dx$$

$$= 0 \qquad \text{Since } g(x) = 0$$

Hence, $$u(x,t) = \frac{8k}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2}\sin\frac{(2m+1)\pi ct}{l}\sin\frac{(2m+1)\pi x}{l}$$

2.Solve the boundary value problem

$$u_{tt} = a^2 u_{xx}; 0 < x < l; t > 0 \; with \; u(0,t) = 0, u(l,t) = 0 \,\&\, u(x,0) = 0, u_t(x,0) = \sin^3\left(\frac{\pi x}{l}\right)$$

Sol.. let $u(x,t)$ is the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} \rightarrow (1)$$

Given conditions are

$$u(0,t) = 0 \forall t \rightarrow (2) \; and \; \mu_t(x,0) = \sin^3\frac{\pi x}{l} \forall x \in [0,l] \rightarrow (5)$$

$$u(l,t) = 0 \forall t \rightarrow (3)$$

$$u(x,0) \forall 0 \le x \le l \rightarrow (4)$$

The required solution of (1) is of the form

$$u(x,t) = (c_1\cos px + c_2\sin px) + (c_3\cos pat + c_4\sin pat) \rightarrow (6)$$

Using (2) & (3), we have

$$c_1 = 0 \; and \; p = \frac{n\pi}{l} \; where \; n = 1,2,3.....$$

$\therefore$ General solution of (1) satisfying (2) & (3) is

$$u(x,t) = c_2\sin\frac{n\pi x}{l}\left(c_3\cos\frac{n\pi at}{l} + c_4\sin\frac{n\pi at}{l}\right) \rightarrow (7)$$

Now using condition (4) $u(x,0) = 0$ we get

$$u(x,0)=0=c_2\sin\frac{n\pi x}{l}(c_3+0)$$

$$\Rightarrow c_2c_3\sin\frac{n\pi x}{l}=0\Rightarrow c_3=0(\because c_2\neq 0)\to(8)$$

from (7) & (8)

$$u(x,t)=c_2\sin\frac{n\pi x}{l}\left(0+c_4\sin\frac{n\pi at}{l}\right)$$

$$=c_n\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}\quad where\ c_n=c_2c_4$$

The most general solution of (1) is

$$u(x,t)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}----(9)$$

$$\frac{\partial(u(x,t))}{\partial t}=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}\left(\frac{n\pi a}{l}\right)$$

$$\frac{\partial u}{\partial t}(x,0)=\sum_{n=1}^{\infty}C_n\frac{n\pi a}{l}\sin\frac{n\pi x}{l}$$

From (5) & above result

$$\sin^3\frac{\pi x}{l}=\sum_{n=1}^{\infty}c_n\frac{n\pi a}{l}\sin\frac{n\pi x}{l}$$

$$\frac{3}{4}\sin\frac{\pi x}{l}-\frac{1}{4}\sin\frac{3\pi x}{l}=\sum_{n=1}^{\infty}c_n\frac{n\pi a}{l}\sin\frac{n\pi x}{l}\left[\because\sin^3\theta=\frac{3}{4}\sin\theta-\frac{1}{4}\sin 3\theta\right]$$

$$=\left[c_1\frac{\pi a}{l}\sin\frac{\pi x}{l}+c_2\frac{2\pi a}{l}\sin\frac{2\pi x}{l}+-----\right]$$

Comparing the coefficients of like terms,

$$c_1\frac{\pi a}{l}=\frac{3}{4},c_2=0,c_3\left(\frac{3\pi a}{l}\right)=\frac{-1}{4},c_4,c_5---c_n=0$$

$$\Rightarrow c_1=\frac{3l}{4\pi a},c_2=0,c_3=\frac{-1}{12\pi a},c_4=0,c_5=0$$

Hence, substituting the values in (9)

$$u(x,t)=\frac{3l}{4\pi a}\sin\frac{\pi x}{l}\sin\frac{\pi at}{l}-\frac{1}{12\pi a}\sin\frac{3\pi x}{l}\sin\frac{3\pi at}{l}$$

3.If a string of length l is initially at rest in equilibrium position and each of its points is given the velocity $V_o\sin^3\frac{\pi x}{l}$, find the displacement $y(x,t)$

Sol. With the explained notation, the displacement $y(x,t)$ is given by

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \to (1)$$

$$y(0,t) = 0 \forall t \to (2)$$

$$y(l,t) = 0 \forall t \to (3)$$

$$y(x,0) = 0, 0 \le x \le l \to (4)$$

$$\left[\frac{\partial y}{\partial t}\right]_{at\, t=0} = V_0 \sin^3 \frac{\pi x}{l} \to (5)$$

The most general solution of (1) satisfying (2) & (3) is

$$y(x,t) = \sum_{n=1}^{\infty}\left(C_n \cos\frac{n\pi ct}{l} + D_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l} \to (6)$$

Using (4) we get $\sum C_n \sin\frac{n\pi x}{l} = 0 \forall x \in [0,l]$ which implies $C_n = 0$ for all n

Now, using (5), we get

$$\sum D_n \frac{n\pi c}{l}\sin\frac{n\pi x}{l} = V_0 \sin^3\frac{\pi x}{l}$$

$$= V_0\left[\frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}\right]$$

Hence $D_1 = \frac{-3l}{4\pi c V_o}, D_3 = \frac{-lV_0}{12\pi c}$

Hence $y(x,t) = \frac{-3lV_o}{4\pi c}\sin\frac{\pi ct}{l}\sin\frac{\pi x}{l} - \frac{lV_0}{12\pi c}\sin\frac{3\pi ct}{l}\sin\frac{3\pi x}{l}$

UNIT V

LAPLACE TRANSFORMS

INTRODUCTION

Laplace Transformations were introduced by Pierre Simmon Marquis De Laplace (1749-1827), a French Mathematician known as a Newton of French. Laplace Transformations is a powerful Technique; it replaces operations of calculus by operations of Algebra. Suppose an Ordinary (or) Partial Differential Equation together with Initial conditions is reduced to a problem of solving an Algebraic Equation.

USES:

- Particular Solution is obtained without first determining the general solution
- Non-Homogeneous Equations are solved without obtaining the complementary Integral
- Solutions of Mechanical (or) Electrical problems involving discontinuous force functions (R.H.S function) (or) Periodic functions other than and are obtained easily.

Applications:

- L.T is applicable not only to continuous functions but also to piece-wise continuous functions, complicated periodic functions, step functions, Impulse functions.

Definition:

Let f (t) be a function of't' defined for all positive values of t. Then Laplace transforms of f (t) is denoted by L {f (t)} is defined by $L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt = \bar{f}(s) \to (1)$

Provided that the integral exists. Here the parameter's' is a real (or) complex number.

The relation (1) can also be written as $f(t) = L^{-1}\{\bar{f}(s)\}$

In such a case the function f(t) is called the inverse Laplace transform of $\bar{f}(s)$.The symbol 'L' which transform f(t) in to $\bar{f}(s)$ is called the Laplace transform operator. The symbol 'L^{-1}' which transforms $\bar{f}$(s) to f (t) can be called the inverse Laplace transform operator.

Conditions for Laplace Transforms

Exponential order: A function f (t) is said to be of exponential order 'a' If $\underset{t\to\infty}{lt}\, e^{-st} f(t) =$ a finite quantity.

Ex: (i). The function t^2 is of exponential order

(ii). The function e^{t^3} is not of exponential order (which is not limit)

Piece – wise Continuous function: A function f (t) is said to be piece-wise continuous over the closed interval [a,b] if it is defined on that interval and is such that the interval can be divided in to a finite number of sub intervals, in each of which f (t) is continuous and has both right and left hand limits at every end point of the subinterval.

Sufficient conditions for the existence of the Laplace transform of a function:

The function f (t) must satisfy the following conditions for the existence of the L.T.

(i).The function f (t) must be piece-wise continuous (or) sectionally continuous in any limited interval $0 < a \le t \le b$

(ii).The function f (t) is of exponential order.

Laplace Transforms of standard functions:

1. Prove that $L\{1\} = \frac{1}{s}$

Proof: By definition

$$L\{1\} = \int_0^\infty e^{-st}.1dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{e^{-\infty}}{-s} - \frac{e^0}{-s} = 0 + \frac{1}{s}\ if\ s > 0$$

$$L\{1\} = \frac{1}{s}\ (\because e^{-\infty} = 0)$$

2. Prove that $L\{t\} = \frac{1}{s^2}$

Proof: By definition

$$L\{t\} = \int_0^\infty e^{-st}.tdt = \left[t.\left(\frac{e^{-st}}{-s}\right) - \int 1.\frac{e^{-st}}{-s}dt\right]_0^\infty$$

$$= \left[t.\frac{e^{-st}}{-s} - \frac{e^{-st}}{(-s)^2}\right]_0^\infty = \frac{1}{s^2}$$

3. Prove that $L\{t^n\} = \frac{n!}{s^{n+1}}$ where n is a +ve integer

Proof: By definition

$$L\{t^n\} = \int_0^\infty e^{-st}.t^n dt = \left[t^n.\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty n.t^{n-1}.\frac{e^{-st}}{-s}dt$$

$$= 0 - 0 + \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}dt$$

$$= \frac{n}{s}L\{t^{n-1}\}$$

$$Similarly \quad L\{t^{n-1}\} = \frac{n-1}{s}L\{t^{n-2}\}$$

$$L\{t^{n-2}\} = \frac{n-2}{s}L\{t^{n-3}\}$$

By repeatedly applying this, we get

$$L\{t^n\} = \frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}.....\frac{2}{s}.\frac{1}{s}L\{t^{n-n}\}$$

$$= \frac{n!}{s^n}L\{1\} = \frac{n!}{s^n}.\frac{1}{s} = \frac{n!}{s^{n+1}}$$

Note: $L\{t^n\}$ can also be expressed in terms of Gamma function.

$$i.e., \mathrm{L}\{t^n\} = \frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}\left(\because \Gamma(n+1) = n!\right)$$

Def: If n>0 then Gamma function is defined by $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$

We have $L\{t^n\} = \int_0^\infty e^{-st}.t^n dt$

Putting x=st on R.H.S, we get

$$L\{t^n\}t = \int_0^\infty e^{-x}.\frac{x^n}{s^n}.\frac{1}{s}dx \qquad \begin{pmatrix} x = st \\ \frac{1}{s}dx = dt \end{pmatrix}$$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-x}.x^n dx \qquad \begin{pmatrix} When & t = 0, x = 0 \\ When & t = \infty, x = \infty \end{pmatrix}$$

$$L\{t^n\} = \frac{1}{s^{n+1}}.\Gamma(n+1)$$

$$If \ 'n' \text{is } a \text{ +ve integer then } \Gamma(n+1) = n!$$

$$\therefore L\{t^n\} = n!/s^{n+1}$$

Note: The following are some important properties of the Gamma function.

1. $\Gamma(n+1) = n.\Gamma(n) \; if \; n > 0$
2. $\Gamma(n+1) = n!$ if n is a +ve integer
3. $\Gamma(1) = 1, \Gamma(1/2) = \sqrt{\pi}$

Note: Value of $\Gamma(n)$ in terms of factorial

$$\Gamma(2) = 1 \times \Gamma(1) = 1!$$
$$\Gamma(3) = 2 \times \Gamma(2) = 2!$$
$$\Gamma(4) = 3 \times \Gamma(3) = 3!$$

In general $\Gamma(n+1) = n!$ provided 'n' is a +ve integer.

Taking n=0, it defined 0! = $\Gamma(1) = 1$

4. Prove that $L\{e^{at}\} = \dfrac{1}{s-a}$

Proof: By definition,

$$L\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \frac{-e^{-\infty}}{s-a} + \frac{e^0}{s-a} = \frac{1}{s-a} \; if \; s > a$$

Similarly $L\{e^{-at}\} = \dfrac{1}{s+a} \; if \; s > -a$

5. Prove that $L\{\sinh at\} = \dfrac{a}{s^2 - a^2}$

Proof: $L\{\sinh at\} = L\left\{\dfrac{e^{at} - e^{-at}}{2}\right\} = \dfrac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\}\right]$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right] = \frac{2a}{2(s^2-a^2)} = \frac{a}{s^2-a^2}$$

6. Prove that $L\{\cosh at\} = \dfrac{s}{s^2 - a^2}$

Proof: $L\{\cosh at\} = L\left\{\dfrac{e^{at} + e^{-at}}{2}\right\}$

$$=\frac{1}{2}\left[L\left\{e^{at}\right\}+L\left\{e^{-at}\right\}\right]=\frac{1}{2}\left\{\frac{1}{s-a}+\frac{1}{s+a}\right\}$$

$$=\frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right]=\frac{2s}{2\left(s^2-a^2\right)}=\frac{s}{s^2-a^2}$$

7. Prove that $L\{\sin at\}=\frac{a}{s^2+a^2}$

Proof: By definition,

$$L\{\sin at\}=\int_0^\infty e^{-st}\sin at\,dt$$

$$=\left[\frac{e^{-st}}{s^2+a^2}(-s\sin at-a\cos at)\right]_0^\infty$$

$$\left[\because\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)\right]$$

$$=\frac{a}{s^2+a^2}$$

8. Prove that $L\{\cos at\}=\frac{s}{s^2+a^2}$

Proof: We know that $L\left\{e^{at}\right\}=\frac{1}{s-a}$

Replace 'a' by 'ia' we get

$$L\left\{e^{iat}\right\}=\frac{1}{s-ia}=\frac{s+ia}{(s-ia)(s+ia)}$$

$$i.e., \mathrm{L}\{\cos at+i\sin at\}=\frac{s+ia}{s^2+a^2}$$

Equating the real and imaginary parts on both sides, we have

$$L\{\cos at\}=\frac{s}{s^2+a^2}\ and\ L\{\sin at\}=\frac{a}{s^2+a^2}$$

Problems

1. Find the Laplace transforms of $(t^2+1)^2$

Sol: Here $f(t)=(t^2+1)^2=t^4+2t^2+1$

$$L\{(t^2+1)^2\}=L\{t^4+2t^2+1\}=L\{t^4\}+2L\{t^2\}+L\{1\}$$

$$=\frac{4!}{s^{4+1}}+2.\frac{2!}{s^3}+\frac{1}{s}=\frac{4!}{s^5}+2.\frac{2!}{s^3}+\frac{1}{s}$$

$$=\frac{24}{s^5}+\frac{4}{s^3}+\frac{1}{s}=\frac{1}{s^5}(24+4s^2+s^4)$$

2. Find the Laplace transform of $L\left\{\frac{e^{-at}-1}{a}\right\}$

Sol: $L\left\{\frac{e^{-at}-1}{a}\right\}=\frac{1}{a}L\left\{e^{-at}-1\right\}=\frac{1}{a}\left[L\left\{e^{-at}\right\}-L\{1\}\right]$

$$=\frac{1}{a}\left[\frac{1}{s+a}-\frac{1}{s}\right]=-\frac{1}{s(s+a)}$$

3. Find the Laplace transform of Sin2tcost

Sol: W.K.T $\sin 2t\cos t=\frac{1}{2}[2\sin 2t\cos t]=\frac{1}{2}[\sin 3t+\sin t]$

$$\therefore L\{\sin 2t\cos t\}=L\left\{\frac{1}{2}[\sin 3t+\sin t]\right\}=\frac{1}{2}\left[\mathrm{L}\{\sin 3t\}+L\{\sin t\}\right]$$

$$=\frac{1}{2}\left[\frac{3}{s^2+9}+\frac{1}{s^2+1}\right]=\frac{2(s^2+3)}{(s^2+1)(s^2+9)}$$

4. Find the Laplace transform of Cosh²2t

Sol: W.K.T $\cosh^2 2t=\frac{1}{2}[1+\cosh 4t]$

$$L\{\cosh^2 2t\}=\frac{1}{2}[L(1)+L\{\cosh 4t\}]$$

$$=\frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2-16}\right]=\frac{s^2-8}{s(s^2-16)}$$

5. Find the Laplace transform of Cos³3t

Sol: Since cos9t=cos3(3t)

$\cos 9t=4\cos^3 3t-3\cos 3t$ (or) $\cos^3 3t=\frac{1}{4}[\cos 9t+3\cos 3t]$

$$L\{\cos^3 3t\}=\frac{1}{4}L\{\cos 9t\}+\frac{3}{4}L\{\cos 3t\}$$

$$\therefore \quad =\frac{1}{4}\cdot\frac{s}{s^2+81}+\frac{3}{4}\cdot\frac{s}{s^2+9}$$

$$=\frac{s}{4}\left[\frac{1}{s^2+81}+\frac{3}{s^2+9}\right]=\frac{s\left(s^2+63\right)}{\left(s^2+9\right)\left(s^2+81\right)}$$

6. Find the Laplace transforms of $\left(\sin t+\cos t\right)^2$

Sol: Since $\left(\sin t+\cos t\right)^2=\sin^2 t+\cos^2 t+2\sin t\cos t \;=1+\sin 2t$

$$L\{(\sin t+\cos t)^2\}=L\{1+\sin 2t\}$$
$$=L\{1\}+L\{\sin 2t\}$$
$$=\frac{1}{s}+\frac{2}{s^2+4}=\frac{s^2+2s+4}{s\left(s^2+4\right)}$$

7. Find the Laplace transforms of cost cos2t cos3t

Sol: $$\cos t\cos 2t\cos 3t=\frac{1}{2}.\cos t\left[2.\cos 2t.\cos 3t\right]$$
$$=\frac{1}{2}\cos t[\cos 5t+\cos t]=\frac{1}{2}[\cos t\cos 5t+\cos^2 t]$$
$$=\frac{1}{4}\left[2\cos t\cos 5t+2\cos^2 t\right]=\frac{1}{4}\left[\left(\cos 6t+\cos 4t\right)+\left(1+\cos 2t\right)\right]$$
$$=\frac{1}{4}[1+\cos 2t+\cos 4t+\cos 6t]$$
$$\therefore L\{\cos t\cos 2t\cos 3t\}=\frac{1}{4}L\{1+\cos 2t+\cos 4t+\cos 6t\}$$
$$=\frac{1}{4}[L\{1\}+L\{\cos 2t\}+L\{\cos 4t\}+L\{\cos 6t\}]$$
$$=\frac{1}{4}\left[\frac{1}{s}+\frac{s}{s^2+4}+\frac{s}{s^2+16}+\frac{s}{s^2+36}\right]$$

8. Find L.T. of Sin^2t

Sol: $$L\{\sin^2 t\}=L\left\{\frac{1-\cos 2t}{2}\right\}$$
$$=\frac{1}{2}[\mathrm{L}\{1\}-L\{\cos 2t\}]=\frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+4}\right]$$

9. Find $\mathbf{L}\left(\sqrt{t}\right)$

Sol: $$L\left\{\sqrt{t}\right\}=L\left[t^{1/2}\right]=\frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}}$$ where n is not an integer
$$=\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}}=\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}\quad \because\Gamma\left(n+1\right)=n.\Gamma\left(n\right)$$

10. Find $L\ \{sin(\omega t+\alpha)\}$, where α a constant is

Sol: $$L\{sin(\omega t+\alpha)\}=L\{sin\omega t cos\alpha+cos\omega t sin\alpha\}$$
$$=cos\alpha\ L\{sin\omega t\}+sin\alpha\ L\{cos\omega t\}$$
$$=cos\alpha\frac{\omega}{s^2+\omega^2}+sin\alpha\frac{\omega}{s^2+\omega^2}$$

Properties of Laplace transform:

Linearity Property:

Theorem1: The Laplace transform operator is a Linear operator.

i.e. $(i).L\{cf(t)\}=c.L\{f(t)\}$ $(ii).L\{f(t)+g(t)\}=L\{f(t)\}+L\{g(t)\}$ Where 'c' is constant

Proof: (i) By definition

$$L\{cf(t)\}=\int_0^\infty e^{-st}cf(t)\,dt=c\int_0^\infty e^{-st}f(t)\,dt=cL\{f(t)\}$$

(ii) By definition

$$L\{f(t)+g(t)\}=\int_0^\infty e^{-st}\{f(t)+g(t)\}dt$$

$$=\int_0^\infty e^{-st}f(t)dt+\int_0^\infty e^{-st}g(t)dt=L\{f(t)\}+L\{g(t)\}$$

Similarly the inverse transforms of the sum of two or more functions of 's' is the sum of the inverse transforms of the separate functions.

Thus, $L^{-1}\{\overline{f}(s)+\overline{g}(s)\}=L^{-1}\{\overline{f}(s)\}+L^{-1}\{\overline{g}(s)\}=f(t)+g(t)$

Corollary: $L\{c_1f(t)+c_2g(t)\}=c_1L\{f(t)\}+c_2L\{g(t)\}$, where c_1, c_2 are constants

Theorem2: If a, b, c be any constants and f, g, h any functions of t, then

$$L\{af(t)+bg(t)-ch(t)\}=a.L\{f(t)\}+b.L\{g(t)\}-cL\{h(t)\}$$

Proof: By the definition

$$L\{af(t)+bg(t)-ch(t)\}=\int_0^\infty e^{-st}\{af(t)+bg(t)-ch(t)\}dt$$

$$=a.\int_0^\infty e^{-st}f(t)\,dt+b\int_0^\infty e^{-st}g(t)\,dt-c\int_0^\infty e^{-st}h(t)\,dt$$

$$=a.L\{f(t)\}+bL\{g(t)\}-cL\{h(t)\}$$

Change of Scale Property:

If $L\{f(t)\}=\overline{f}(s)$ *then* $L\{f(at)\}=\frac{1}{a}.\overline{f}\left(\frac{s}{a}\right)$

Proof: By the definition we have

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)dt$$

$$Put \quad at = u \Rightarrow dt = \frac{du}{a}$$

$$when \ \ t \to \infty \ \ then \ \ u \to \infty \ \ and \ t = 0 \ \ then \ u = 0$$

$$\therefore L\{f(at)\} = \int_0^\infty e^{-\frac{su}{a}} f(u)\frac{du}{a} = \frac{1}{a}.\int_0^\infty e^{-\left(\frac{s}{a}\right).u} f(u)du = \frac{1}{a}.\overline{f}\left(s/a\right)$$

1. Find $L\{\sinh 3t\}$

Sol: $L\{\sinh t\} = \frac{1}{s^2-1} = \overline{f}(s)$

$\therefore L\{\sinh 3t\} = \frac{1}{3}\overline{f}(s/3)$(Change of scale property)

$$= \frac{1}{3}\frac{1}{\left(s/3\right)^2 - 1} = \frac{3}{s^2-9}$$

2. Find $L\{\cos 7t\}$

Sol: $L\{\cos t\} = \frac{s}{s^2+1} = \overline{f}(s)\ (say)$

$L\{\cos 7t\} = \frac{1}{7}\overline{f}(s/7)$ (Change of scale property)

$$L\{\cos 7t\} = \frac{1}{7}\frac{s/7}{(s/7)^2+1} = \frac{s}{s^2+49}$$

First shifting property:

If $L\{f(t)\} = \overline{f}(s)$ then $L\{e^{at} f(t)\} = \overline{f}(s - a)$

Proof: By the definition

$$L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)dt$$

$$= \int_0^\infty e^{-ut} f(t)dt \ where\, u = s - a$$

$$= \overline{f}(u) = \overline{f}(s-a)$$

Note: Using the above property, we have $L\{e^{-at} f(t)\} = \overline{f}(s+a)$

Applications of this property, we obtain the following results

1. $L\{e^{at} t^n\} = \frac{n!}{(s-a)^{n+1}} \left[\because L(t^n) = \frac{n!}{s^{n+1}}\right]$

2. $L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}\left[\because L(\sin bt) = \frac{b}{s^2+b^2}\right]$

3. $L\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}\left[\because L(\cos bt) = \frac{s}{s^2+b^2}\right]$

4. $L\{e^{at}\sinh bt\} = \frac{b}{(s-a)^2-b^2}\left[\because L(\sinh bt) = \frac{b}{s^2-b^2}\right]$

5. $L\{e^{at}\cosh bt\} = \frac{s-a}{(s-a)^2-b^2}\left[\because L(\cosh bt) = \frac{s}{s^2-b^2}\right]$

1. Find the Laplace Transforms of t^3e^{-3t}

Sol: Since $L\{t^3\} = \frac{3!}{s^4}$

Now applying first shifting theorem, we get

$$L\{t^3e^{-3t}\} = \frac{3!}{(s+3)^4}$$

2. Find the L.T. of $e^{-t}\cos 2t$

Sol: Since $L\{\cos 2t\} = \frac{s}{s^2+4}$

Now applying first shifting theorem, we get

$$L\{e^{-t}\cos 2t\} = \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}$$

3. Find L.T of $e^{2t}cos^2t$

Sol: - $L\,[e^{2t}cos^2t] = L\,[e^{2t}(\frac{1+cos2t}{2})]$

$$=\frac{1}{2}\{L[e^{2t}] + L[e^{2t}cos2t]\}$$

$$=\frac{1}{2}(\frac{1}{s-2})+\frac{1}{2}\{L[cos2t]\}_{s\to s-2}$$

$$=\frac{1}{2}(\frac{1}{s-2})+\frac{1}{2}\frac{s-2}{(s-2)^2+2^2}$$

$$=\frac{1}{2}(\frac{1}{s-2})+\frac{1}{2}\frac{s-2}{(s^2-4s+8)}$$

Second translation (or) second Shifting theorem:

If $L\{f(t)\} = \overline{f}(s) and\ g(t) = \begin{cases} f(t-a), & t>a \\ 0 & t<a \end{cases} then\ L\{g(t)\} = e^{-as}\overline{f}(s)$

Proof: By the definition

$$L\{g(t)\} = \int_0^\infty e^{-st}\,g(t)dt = \int_0^a e^{-st}\,g(t)dt + \int_a^\infty e^{-st}\,g(t)dt$$

$$= \int_0^\infty e^{-st}.\,odt + \int_a^\infty e^{-st}\,f(t-a)dt = \int_a^\infty e^{-st}\,f(t-a)dt$$

Let t-a = u so that dt = du And also u = 0 when t = a and $u \to \infty$ when $t \to \infty$

$\therefore L\{g(t)\} = \int_0^{\infty} e^{-s(u+a)} f(u)du = e^{-as} \int_0^{\infty} e^{-su} f(u)du = e^{-as} \int_a^{\infty} e^{-st} f(t)dt$

$= e^{-as} L\{f(t)\} = e^{-as}\overline{f}(s)$

Another Form of second shifting theorem:

If $L\{f(t)\} = \overline{f}(s)$ and $a > 0$ then $L\{F(t-a)H(t-a)\} = e^{-as}\overline{f}(s)$

where H (t) = $\begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$ and H(t) is called Heaviside unit step function.

Proof: By the definition

$$L\{F(t-a)H(t-a)\} = \int_0^{\infty} e^{-st} F(t-a)H(t-a)dt \to (1)$$

Put t-a=u so that dt= du and also when t=0, u=-a when t $\to \infty$, u$\to \infty$

Then $L\{F(t-a)H(t-a)\} = \int_a^{\infty} e^{-s(u+a)} F(u)H(u)du.$ [by eq(1)]

$$= \int_{-a}^{0} e^{-s(u+a)} F(u)H(u)du + \int_0^{\infty} e^{-s(u+a)} F(u)H(u)du$$

$$= \int_{-a}^{0} e^{-s(u+a)} F(u).0du + \int_0^{\infty} e^{-s(u+a)} F(u).1du$$

[Since By the definition of H (t)]

$$= \int_0^{\infty} e^{-s(u+a)} F(u)du = e^{-as} \int_a^{\infty} e^{-su} F(u)du$$

$= e^{-sa}\int_0^{\infty} e^{-st} F(t)dt$ by property of Definite Integrals

$$= e^{-as} L\{F(t)\} = e^{-as}\overline{f}(s)$$

Note: $H(t-a)$ *is also denoted by* $u(t-a)$

1. **Find the L.T. of g (t) when** $g(t) = \begin{cases} \cos\left(t - \pi/3\right) & if\ t > \pi/3 \\ 0 & if\ t < \pi/3 \end{cases}$

Sol. Let $f(t) = \cos t$

$\therefore L\{F(t)\} = L\{cost\} = \frac{s}{s^2+1} = \overline{f}(s)$

$$g(t) = \begin{cases} f\left(t - \pi/3\right) = cos\left(t - \pi/3\right), & if\ t > \pi/3 \\ 0 & ,\ if\ t < \pi/3 \end{cases}$$

Now applying second shifting theorem, then we get

$$L\{g(t)\} = e^{\frac{-\pi s}{3}}\left(\frac{s}{s^2+1}\right) = \frac{s.e^{\frac{-\pi s}{3}}}{s^2+1}$$

2. **Find the L.T. of (ii)$(t-2)^3 u(t-2)$ (ii) $e^{-3t}u(t-2)$**

Sol:

(i). Comparing the given function with f(t-a) u(t-a), we have a=2 and f(t)=t^3

$$\therefore L\{f(t)\} = L\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4} = \overline{f}(s)$$

Now applying second shifting theorem, then we get

$$L\{(t-2)^3 u(t-2)\} = e^{-2s}\frac{6}{s^4} = \frac{6e^{-2s}}{s^4}$$

(ii). $L\{e^{-st}u(t-2)\} = L\{e^{-s(t-2)}.e^{-6}u(t-2)\} = e^{-6}L\{e^{-3(t-2)}u(t-2)\}$

$f(t) = e^{-3t}$ then $\overline{f}(s) = \frac{1}{s+3}$

Now applying second shifting theorem then, we get

$$L\{e^{-3t}u(t-2)\} = e^{-6}.e^{-2s}\frac{1}{s+3} = \frac{e^{-2(s+3)}}{s+3}$$

Multiplication by't':

Theorem: If $L\{f(t)\} = \overline{f}(s)$ ***then*** $L\{tf(t)\} = \frac{-d}{ds}\overline{f}(s)$

Proof: By the definition $\overline{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$

$$\frac{d}{ds}\{\overline{f}(s)\} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

By Leibnitz's rule for differentiating under the integral sign,

$$\frac{d}{ds}\overline{f}(s) = \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\,dt$$

$$= \int_0^\infty -te^{-st} f(t)\,dt$$

$$= -\int_0^\infty e^{-st}\{tf(t)\}dt = -L\{tf(t)\}$$

Thus $L\{tf(t)\} = \frac{-d}{ds}\overline{f}(s)$

$$\therefore L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} = \overline{f}(s)$$

Note: Leibnitz's Rule

If $f(x,\alpha)$ *and* $\frac{\partial}{\partial \alpha} f(x,\alpha)$ be continuous functions of x and α then

$$\frac{d}{d\alpha}\left\{\int_a^b f(x,\alpha)dx\right\} = \int_a^b \frac{\partial}{\partial \alpha} f(x,\alpha)dx$$

Where a, b are constants independent of α

Problems:

1. Find L.T of tcosat

Sol: Since $L\{tcos\ at\} = \frac{s}{s^2+a^2}$

$$L\{tcos\ at\} = -\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$

$$= \frac{-s^2+a^2-s.2s}{(s^2+a^2)^2} = \frac{s^2-a^2}{(s^2+a^2)^2}$$

2. Find t^2sin at

Sol: Since $L\{sin\ at\} = \frac{a}{s^2+a^2}$

$$L\{t^2.sin\ at\} = (-1)^2\frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{d}{ds}\left(\frac{-2as}{\left(s^2+a^2\right)^2}\right) = \frac{2a\left(3s^2-a^2\right)}{\left(s^2+a^2\right)^3}$$

3. Find $L.T\ of\ te^{-t}\sin 3t$

Sol: Since $L\{sin\ 3t\} = \frac{3}{s^2+3^2}$

$$\therefore L\{tsin\ 3t\} = \frac{-d}{ds}\left[\frac{3}{s^2+3^2}\right] = \frac{6s}{(s^2+9)^2}$$

Now using the shifting property, we get

$$L\{te^{-t}sin\ 3t\} = \frac{6(s+1)}{((s+1)^2+9)^2} = \frac{6(s+1)}{(s^2+2s+10)^2}$$

4. Find $L\{te^{2t}sin\ 3t\}$

Sol: Since $L\{sin\ 3t\} = \frac{3}{s^2+9}$

$$\therefore L\left\{e^{2t}\sin 3t\right\} = \frac{3}{\left(s-2\right)^2+9} = \frac{3}{s^2-4s+13}$$

$$L\{te^{2t}sin\ 3t\} = (-1)\frac{d}{ds}\left[\frac{3}{s^2-4s+13}\right] = (-1)\left[\frac{0-3(2s-4)}{(s^2-4s+13)^2}\right]$$

$$= \frac{3(2s-4)}{\left(s^2-4s+13\right)^2} = \frac{6(s-2)}{\left(s^2-4s+13\right)^2}$$

5. Find the L.T. of $\left(1+te^{-t}\right)^2$

Sol: Since $(1+te^{-t})^2 = 1+2te^{-t}+t^2e^{-2t}$

$$\therefore L\left(1+te^{-t}\right)^2 = L\{1\}+2L\left\{te^{-t}\right\}+L\left\{t^2e^{-2t}\right\}$$

$$= \frac{1}{s}+2(-1)\frac{d}{ds}\left(\frac{1}{s+1}\right)+(-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s+2}\right)$$

$$=\frac{1}{s}+\frac{2}{(s+1)^2}+\frac{d}{ds}\left(\frac{-1}{(s+2)^2}\right)$$

$$=\frac{1}{s}+\frac{2}{(s+1)^2}+\frac{2}{(s+2)^3}$$

6. Find the L.T of t^3e^{-3t} (already we have solved by another method)

Sol: $L\{t^3e^{-3t}\}=(-1)^3\frac{d^3}{ds^3}L\{e^{-3t}\}$

$$=-\frac{d^3}{ds^3}\left(\frac{1}{s+3}\right)=\frac{-3!(-1)^3}{(s+3)^4}$$

$$=\frac{3!}{(s+3)^4}$$

7. Find $L\{\cosh at \sin at\}$

Sol. $L\{\cosh at \sin at\}=L\left\{\frac{e^{at}+e^{-at}}{2}.\sin at\right\}$

$$=\frac{1}{2}[L\{e^{at}\sin at\}+L\{e^{-at}\sin at\}]$$

$$=\frac{1}{2}\left[\frac{a}{(s-a)^2+a^2}+\frac{a}{(s+a)^2+a^2}\right]$$

8. Find the L.T of the function $f(t)=(t-1)^2, \quad t>1$
$=0 \quad 0<t<1$

Sol: By the definition

$$L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt=\int_0^1 e^{-st}f(t)dt+\int_1^\infty e^{-st}f(t)dt$$

$$=\int_0^1 e^{-st}0dt+\int_1^\infty e^{-st}(t-1)^2dt$$

$$=\int_1^\infty e^{-st}(t-1)^2dt=\left[(t-1)^2\frac{e^{-st}}{-s}\right]_1^\infty-\int_1^\infty 2(t-1)\frac{e^{-st}}{-s}dt$$

$$=0+\frac{2}{s}\int_1^\infty e^{-st}(t-1)dt$$

$$=\frac{2}{s}\left[\left\{(t-1)\left(\frac{e^{-st}}{-s}\right)\right\}_1^\infty-\int_1^\infty\frac{e^{-st}}{-s}dt\right]$$

$$=\frac{2}{s}\left[0+\frac{1}{s}\int_1^\infty e^{-st}dt\right]=\frac{2}{s^2}\left(\frac{e^{-st}}{-s}\right)_1^\infty=\frac{-2}{s^3}\left(e^{-st}\right)_1^\infty$$

$$=\frac{-2}{s^3}\left(0-e^{-s}\right)=\frac{2}{s^3}e^{-s}$$

9. **Find the L.T of f (t) defined as** $f(t)=3$, **t >2**

$=0$, **0<t<2**

Sol: $L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$

$$=\int_0^2 e^{-st} f(t)dt + \int_2^\infty e^{-st} f(t)dt$$

$$= \int_0^2 e^{-st}.0dt + \int_2^\infty e^{-st}\,3dt$$

$$=0+\int_2^\infty e^{-st}3dt = \frac{-3}{s}\left(e^{-st}\right)_2^\infty = \frac{-3}{s}\left(0-e^{-2s}\right)$$

$$=\frac{3}{s}e^{-2s}$$

10. **Find** $L\{t\,cos(at+b)\}$

Sol: $L\{cos(at+b)\} = L\{\cos at \cos b - \sin at \sin b\}$

$$= \cos b. L\{\cos at\} - \sin b\, L\{\sin at\}$$

$$= \cos b.\frac{s}{s^2+a^2} - \sin b.\frac{a}{s^2+a^2}$$

$$L\{t.cos(at+b)\} = \frac{-d}{ds}\left[\cos b.\frac{s}{s^2+a^2} - \sin b.\frac{a}{s^2+a^2}\right]$$

$$= -\cos b.\left(\frac{s^2+a^2.1-s.2s}{\left(s^2+a^2\right)^2}\right) + \sin b\left(\frac{\left(s^2+a^2\right).0-a.2s}{\left(s^2+a^2\right)^2}\right)$$

$$=\frac{1}{\left(s^2+a^2\right)^2}\left[\left(s^2-a^2\right)^2\cos b - 2as\sin b\right]$$

11. **Find L.T of L** $[te^t sint]$

Sol: - We know that $L[sint] = \frac{1}{s^2+1}$

$$L[tsint] = (-1)\frac{d}{ds}L[sint] = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = -\frac{(-1)2s}{(s^2+1)^2}$$

$$=\frac{2s}{(s^2+1)^2}$$

By First Shifting Theorem

$$L\,[te^t sint] = \left[\frac{2s}{(s^2+1)^2}\right]_{s\to s-1} = \frac{2(s-1)}{((s-1)^2+1)^2} = \frac{2(s-1)}{(s^2-2s+2)^2}$$

Division by't':

Theorem: If $L\{f(t)\} = \overline{f}(s)$ **then** $L\left\{\frac{1}{t}f(t)\right\} = \int_s^\infty \overline{f}(s)ds$

Proof: We have $\overline{f}(s)=\int_0^\infty e^{-st} f(t)dt$

Now integrating both sides w.r.t s from s to ∞, we have

$$\int_0^\infty \overline{f}(s)ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)dt\right]ds$$

$$= \int_0^\infty \int_s^\infty f(t)e^{-st}dsdt \text{ (Change the order of integration)}$$

$$= \int_0^\infty f(t)\left[\int_s^\infty e^{-st}ds\right]dt \text{ (} \because \text{ t is independent of‘s’)}$$

$$= \int_0^\infty f(t)\left(\frac{e^{-st}}{-t}\right)_s^\infty dt$$

$$= \int_0^\infty e^{-st}\frac{f(t)}{t}dt (or) \mathcal{L}\left\{\frac{1}{t}f(t)\right\}$$

Problems:

1. **Find** $L\left\{\frac{\sin t}{t}\right\}$

Sol: Since $L\{sint\} = \frac{1}{s^2+1} = \overline{f}(s)$

Division by‘t’, we have

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \overline{f}(s)ds = \int_s^\infty \frac{1}{s^2+1}ds$$

$$= [Tan^{-1}s]_s^\infty = Tan^{-1}\infty - Tan^{-1}s$$

$$= \pi/2 - Tan^{-1}s = \cot^{-1}s$$

2. **Find the L.T of** $\frac{\sin at}{t}$

Sol: Since $L\{\sin at\} = \frac{a}{s^2+a^2} = \overline{f}(s)$

Division by t, we have

$$L\left\{\frac{\sin a\, t}{t}\right\} = \int_s^\infty \overline{f}(s)ds = \int_s^\infty \frac{a}{s^2+a^2}ds$$

$$= a.\frac{1}{a}\left[Tan^{-1}\, s/a\right]_s^\infty = Tan^{-1}\infty - Tan^{-1}\, s/a$$

$$= \pi/2 - Tan^{-1}(s/a) = \cot^{-1} s/a$$

3. **Evaluate** $L\left\{\frac{1-\cos a\, t}{t}\right\}$

Sol: Since $L\{1-\cos a\, t\} = L\{1\} - L\{cos\ at\} = \frac{1}{s} - \frac{s}{s^2+a^2}$

$$L\left\{\frac{1-\cos a\, t}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+a^2}\right)ds$$

$$= \left[\log s - \frac{1}{2}\log\left(s^2+a^2\right)\right]_s^\infty$$

$$=\frac{1}{2}\left[2\log s-\log\left(s^2+a^2\right)\right]_s^\infty=\frac{1}{2}\left[\log\left(\frac{s^2}{s^2+a^2}\right)\right]_s^\infty$$

$$=\frac{1}{2}\left[l\log\left(\frac{1}{1+a^2/s^2}\right)\right]_s^\infty=\frac{1}{2}\left[\log 1-\log\frac{s^2}{s^2+a^2}\right]$$

$$=-\frac{1}{2}l\log\left(\frac{s^2}{s^2+a^2}\right)=\log\left(\frac{s^2}{s^2+a^2}\right)^{\frac{-1}{2}}=\log\sqrt{\frac{s^2+a^2}{s^2}}$$

Note: $L\left\{\frac{1-\cos t}{t}\right\}=\log\sqrt{\frac{s^2+1}{s}}$ (Putting a=1 in the above problem)

4. Find $L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}$

Sol: $L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}=\int_s^\infty\left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds$

$$=\left[\log(s+a)-\log(s+b)\right]_s^\infty=\left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$$

$$=\underset{s\to\infty}{lt}\left\{\log\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right\}-\log\left(\frac{s+a}{s+b}\right)$$

$$=\log 1-\log(s+a)+\log(s+b)=\log\left(\frac{s+b}{s+a}\right)$$

5. Find $L\left\{\frac{1-\cos t}{t^2}\right\}$

Sol: $L\left\{\frac{1-\cos t}{t^2}\right\}=L\left\{\frac{1}{t}.\frac{1-\cos t}{t}\right\}$.....(1)

Now $L\left\{\frac{1-\cos t}{t}\right\}=\int_s^\infty\left(\frac{1}{s}-\frac{s}{s^2+1}\right)ds=\left[\log s-\frac{1}{2}\log\left(s^2+1\right)\right]_s^\infty$

$$=\frac{1}{2}\left[\log\frac{s^2}{s^2+1}\right]_s^\infty=\frac{-1}{2}\left[\log\frac{s^2}{s^2+1}\right]=\frac{1}{2}\log\frac{s^2+1}{s^2}$$

$$\therefore L\left[\frac{1-\cos t}{t^2}\right]=\int_s^\infty\frac{1}{2}\log\frac{s^2+1}{s^2}ds$$

$$=\frac{1}{2}\left[\left\{\log\left(\frac{s^2+1}{s^2}\right)\right\}.s\right]_s^\infty-\int_s^\infty\frac{s^2}{s^2+1}\left(\frac{-2}{s^3}\right).sds$$

$$=\frac{1}{2}\left[\left\{\underset{s\to\infty}{lt}\ s.\log\left(1+\frac{1}{s^2}\right)\right\}-s\log\left(\frac{s^2+1}{s^2}\right)+2\int_s^\infty\frac{ds}{s^2+1}\right]$$

$$=\frac{1}{2}\left[\left\{\underset{s\to\infty}{lt}\ s\left(\frac{1}{s^2}-\frac{1}{2s^4}+\frac{1}{3s^6}+....\right)-s\log\frac{s^2+1}{s^2}\right\}+2Tan^{-1}s\right]_s^\infty$$

$$=\frac{1}{2}\left[\left\{0-s\log\left(1+\frac{1}{s^2}\right)+2\left(\frac{\pi}{2}-Tan^{-1}s\right)\right\}\right]\quad\because\left(\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+.....\right)$$

$$=\cot^{-1}s-\frac{1}{2}s\log\left(1+\frac{1}{s^2}\right)$$

3. Find L.T of $\frac{e^{-at}-e^{-bt}}{t}$

Sol: W.K.T L $[e^{-at}]=\frac{1}{s+a}$, L $[e^{-bt}]=\frac{1}{s+b}$

$$L[\frac{f(t)}{t}]=\int_s^\infty \bar{f}(s)ds$$

$$\therefore\ L\left[\frac{e^{-at}-e^{-bt}}{t}\right]=\int_s^\infty(\frac{1}{s+a}-\frac{1}{s+b})ds$$

$$=[\log(s+a)-\log(s+b)]_s^\infty$$

$$=\log(\frac{s+a}{s+b})_s^\infty$$

$$=\log(\frac{1+\frac{a}{s}}{1+\frac{b}{s}})_s^\infty$$

$$=\log(1)-\log(\frac{s+a}{s+b})$$

$$=0-\log(\frac{s+a}{s+b})=\log(\frac{s+b}{s+a})$$

Laplace transforms of Derivatives:

If $f^1(t)$ be continuous and $L\{f(t)\}=\bar{f}(s)$ then $L\{f^1(t)\}=s\bar{f}(s)-f(0)$

Proof: By the definition

$$L\{f^1(t)\}=\int_0^\infty e^{-st}f^1(t)dt$$

$$=\left[e^{-st}f(t)\right]_0^\infty-\int_0^\infty(-s)e^{-st}f(t)dt \qquad \text{(Integrating by parts)}$$

$$=\left[e^{-st}f(t)\right]_0^\infty+s\int_0^\infty e^{-st}f(t)dt$$

$$=\underset{t\to\infty}{lt}\ e^{-st}f(t)-f(0)+s.L\{f(t)\}$$

Since f (t) is exponential order

$$\therefore \underset{t\to\infty}{lt}\ e^{-st}f(t)=0$$

$$\therefore L\{f^1(t)\}=0-f(0)+sL\{f(t)\}$$

$$= s\overline{f}(s) - f(0)$$

The Laplace Transform of the second derivative $f^{11}(t)$ is similarly obtained.

$$\therefore L\{f^{11}(t)\} = s.L\{f^{1}(t)\} - f^{1}(0)$$

$$= s.\left[s\overline{f}(s) - f(0)\right] - f^{1}(0)$$

$$= s^{2}\overline{f}(s) - sf(0) - f^{1}(0)$$

$$\therefore L\{f^{111}(t)\} = s.L\{f^{11}(t)\} - f^{11}(0)$$

$$= s[s^{2}L\{f(t)\} - sf(0) - f^{1}(0)] - f^{11}(0)$$

$$= s^{3}L\{f(t)\} - s^{2}f(0) - sf^{1}(0) - f^{11}(0)$$

Proceeding similarly, we have

$$L\{f^{n}(t)\} = s^{n}L\{f(t)\} - s^{n-1}f(0) - s^{n-2}f^{1}(0) \ldots\ldots f^{n-1}(0)$$

Note 1: $L\{f^{n}(t)\} = s^{n}\overline{f}(s)\ if\ f(0) = 0\ and\ f^{1}(0) = 0, f^{11}(0) = 0 \ldots\ f^{n-1}(0) = 0$

Note 2: Now $|f(t)| \le M.e^{at}\ for\ all\ t \ge 0\ and\ for\ some\ constants\ a\ amd\ M.$

We have $|e^{-st}f(t)| = e^{-st}|f(t)| \le e^{at}.Me^{at}$

$=$M. $e^{-(s-a)t} \to 0\ as\ t\ \to \infty$ if s>a

$$\therefore \underset{t\to\infty}{lt}\ e^{-st}f(t) = 0\ for\ \ s > a$$

Problems:

Using the theorem on transforms of derivatives, find the Laplace Transform of the following functions.

(i). e^{at} (ii). cosat (iii). t sin at

(i). $Let\ f(t) = e^{at}\ Then\ f^{1}(t) = a.e^{at}\ and\ f(0) = 1$

$$Now\ L\{f^{1}(t)\} = s.L\{f(t)\} - f(0)$$

$$i.e., L\{ae^{at}\} = s.L\{e^{at}\} - 1$$

$$i.e., L\{e^{at}\} - s.L\{e^{at}\} = -1$$

$$i.e., (a - s)L\{e^{at}\} = -1$$

$$\therefore L\{e^{at}\} = \frac{1}{s-a}$$

(ii). $Let\ f(t) = cosat\ then\ f^{1}(t) = -asinat\ and\ f^{11}(t) = -a^{2}cosat$

$$\therefore L\{f^{11}(t)\} = s^{2}L\{f(t)\} - s.f(0) - f^{1}(0)$$

$$Now\ f(0) = \cos 0 = 1\ and\ \ f^{1}(0) = -a\sin 0 = 0$$

$$Then\ L\{-a^{2}\cos at\} = s^{2}L\{\cos at\} - s.1 - 0$$

$$\Rightarrow -a^{2}L\{\cos at\} - s^{2}L\{\cos at\} = -s$$

$$\Longrightarrow -(s^2+a^2)L\{\cos at\} = -s \Rightarrow L\{\cos at\} = \frac{s}{s^2+a^2}$$

(iii). $Let\ f(t) = t\sin at\ then\ f^1(t) = \sin at + at\cos at$

$$f^{11}(t) = a\cos at + a[\cos at - at\sin at] = 2a\cos at - a^2 t\sin at$$

Also $f(0) = 0$ and $f^1(0) = 0$

$Now\ L\{f^{11}(t)\} = s^2L\{f(t)\} - sf(0) - f^1(0)$

$i.e., L\{2a\cos at - a^2 t\sin at\} = s^2L\{t\sin at\} - 0 - 0$

$i.e., 2a\,L\{\cos at\} - a^2L\{t\sin at\} - s^2L\{t\sin at\} = 0$

$i.e., -(s^2+a^2)L\{t\sin at\} = \frac{-2as}{s^2+a^2} \Rightarrow L\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}$

Laplace Transform of Integrals:

If $L\{f(t)\} = \overline{f}(s)\ then\ L\left\{\int_0^t f(x)\,dx\right\} = \frac{\overline{f}(s)}{s}$

Proof: $Let\ g(t) = \int_0^t f(x)dx$

Then $g^1(t) = \frac{d}{dt}\left[\int_0^t f(x)\,dx\right] = f(t)\ and\ g(0) = 0$

Taking Laplace Transform on both sides

$L\{g^1(t)\} = L\{f(t)\}$

But $L\{g^1(t)\} = sL\{g(t)\} - g(0) = sL\{g(t)\} - 0$ $[Since\ g(0) = 0]$

$\therefore L\{g^1(t)\} = L\{f(t)\}$

$\Longrightarrow sL\{g(t)\} = L\{f(t)\} \Rightarrow L\{g(t)\} = \frac{1}{s}L\{f(t)\}$

$But\ g(t) = \int_0^t f(x)\,dx$

$\therefore L\left\{\int_0^t f(x)\,dx\right\} = \frac{\overline{f}(s)}{s}$

Problems:

1. Find the L.T of $\int_0^t \sin at\,dt$

Sol: $L\{\sin at\} = \frac{a}{s^2+a^2} = \overline{f}(s)$

Using the theorem of Laplace transform of the integral, we have

$L\left\{\int_0^t f(x)\,dx\right\} = \frac{\overline{f}(s)}{s}$

$\therefore L\left\{\int_0^t \sin at\right\} = \frac{a}{s(s^2+a^2)}$

2. Find the L.T of $\int_0^t \frac{\sin t}{t}dt$

Sol: $L\{\sin t\} = \frac{1}{s^2+1}\ also\ \underset{t\to 0}{lt}\ \frac{\sin t}{t} = 1\ exists$

$$\therefore L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\}ds = \int_s^\infty \frac{1}{s^2+1}ds$$

$$= \left[Tan^{-1}s\right]_s^\infty = Tan^{-1}\infty - Tan^{-1}s = \pi/2 - Tan^{-1}s = \cot^{-1}s\,(or)\,Tan^{-1}\left(1/s\right)$$

$$i.e., L\left\{\frac{\sin t}{t}\right\} = Tan^{-1}(1/s)(or)cot^{-1}s$$

$$\therefore\ L\left\{\int_0^t \frac{\sin t}{t}dt\right\} = \frac{1}{s}Tan^{-1}(1/s)\ (or)\ \frac{1}{s}cot^{-1}s$$

3. **Find L.T of** $e^{-t}\int_0^t \frac{sint}{t}dt$

Sol: L [$e^{-t}\int_0^t \frac{sint}{t}dt$]

We know that

$$\text{L}\,\{\text{sint}\} = \frac{1}{s^2+1} = \bar{f}(s)$$

$$\text{L}\,\{\frac{sint}{t}\} = \int_s^\infty \bar{f}(s)ds = \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$=(tan^{-1}s)_s^\infty$$

$$=tan^{-1}\infty - tan^{-1}s = \frac{\pi}{2} - tan^{-1}s\ = cot^{-1}s$$

$$\therefore\ \text{L}\left\{\frac{sint}{t}\right\} = cot^{-1}s$$

Hence $\text{L}\,\{\int_0^t \frac{sint}{t}dt\} = \frac{1}{s}cot^{-1}s$

By First Shifting Theorem

$$\text{L}\,[\,e^{-t}\int_0^t \frac{sint}{t}dt] = \bar{f}(s+1) = (\frac{cot^{-1}s}{s})_{s\to s+1}$$

$$\therefore\ \text{L}\left[e^{-t}\int_0^t \frac{sint}{t}dt\right] = \frac{1}{s+1}cot^{-1}(s+1)$$

Laplace transform of Periodic functions:

If f (t) is a periodic function with period 'a'. i.e, $f(t+a)=f(t)$ then

$$\text{L}\ \{f(t)\} = \frac{1}{1-e^{-sa}}\int_0^a e^{-st}f(t)dt$$

Eg: sin x is a periodic function with period 2π

$i.e.,\ \sin x = \sin(2\pi + x) = \sin(4\pi + x)$............

Problems:

1. **A function f (t) is periodic in (0,2b) and is defined as** $f(t) \begin{array}{l} =1\ if\ \ 0<t<b \\ =-1\ if\ \ b<t<2b \end{array}$

Find its Laplace Transform.

Sol: $$L\{f(t)\}=\frac{1}{1-e^{-2bs}}\int_0^{2b}e^{-st}f(t)\,dt$$

$$=\frac{1}{1-e^{-2bs}}\left[\int_0^{b}e^{-st}f(t)\,dt+\int_b^{2b}e^{-st}f(t)\,dt\right]$$

$$=\frac{1}{1-e^{-2bs}}\left[\int_0^{b}e^{-st}dt-\int_b^{2b}e^{-st}dt\right]$$

$$=\frac{1}{1-e^{-2bs}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{b}-\left(\frac{e^{-st}}{-s}\right)_b^{2b}\right]$$

$$=\frac{1}{s\left(1-e^{-2bs}\right)}\left[-\left(e^{-sb}-1\right)+\left(e^{-2bs}-e^{-sb}\right)\right]$$

$$L\{f(t)\}=\frac{1}{s\left(1-e^{-2bs}\right)}\left[1-2e^{-sb}+e^{-2bs}\right]$$

2. Find the L.T of the function $f(t)=\sin\omega t \;\; if\; 0<t<\frac{\pi}{\omega}$

$$=0\; if\; \frac{\pi}{\omega}<t<\frac{2\pi}{\omega}\; where\; f(t)\; has\; period\; \frac{2\pi}{\omega}$$

Sol: Since f (t) is a periodic function with period $\frac{2\pi}{\omega}$

$$L\{f(t)\}=\frac{1}{1-e^{-sa}}\int_0^{a}e^{-st}f(t)dt$$

$$L\{f(t)\}=\frac{1}{1-e^{-s2\pi/\omega}}\int_0^{2\pi/\omega}e^{-st}f(t)dt$$

$$=\frac{1}{1-e^{-2s\pi/\omega}}\left[\int_0^{\pi/\omega}e^{-st}\sin\omega t\,dt+\int_{\pi/\omega}^{2\pi/\omega}e^{-st}.0dt\right]$$

$$=\frac{1}{1-e^{-2s\pi/\omega}}\left[\frac{e^{-st}\left(-s\sin\omega t-\omega\cos\omega t\right)}{s^2+\omega^2}\right]_0^{\pi/\omega}$$

$$\because\int_a^{b}e^{at}\sin bt=\frac{e^{at}}{a^2+b^2}\left(a\sin bt-b\cos bt\right)$$

$$=\frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{1}{s^2+\omega^2}\left(e^{-s\pi/\omega}.\omega+\omega\right)\right]$$

Laplace Transform of Some special functions:

1. The Unit step function or Heaviside's Unit functions:

It is defined as $u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$

Laplace Transform of unit step function:

To prove that $L\{u(t-a)\} = \frac{e^{-as}}{s}$

Proof: Unit step function is defined as $u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$

Then $L\{u(t-a)\} = \int_0^\infty e^{-st}u(t-a)\,dt$

$$= \int_0^a e^{-st}u(t-a)dt + \int_a^\infty e^{-st}u(t-a)dt$$

$$= \int_0^a e^{-st}.0dt + \int_a^\infty e^{-st}.1dt$$

$$= \int_a^\infty e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty = -\frac{1}{s}.\left[e^{-\infty} - e^{-as}\right] = \frac{e^{-as}}{s}$$

$\therefore L\{u(t-a)\} = \frac{e^{-as}}{s}$

Laplace Transforms of Dirac Delta Function:

The Dirac delta function or Unit impulse function $f_\in(t) = \begin{cases} 1/\in & 0 \le t \le \in \\ 0 & t > \in \end{cases}$

2. Prove that $L\{f_\in(t)\} = \frac{1-e^{-s\in}}{s\in}$ hence show that $L\{\delta(t)\} = 1$

Proof: By the definition $f_\in(t) = \begin{cases} 1/\in & 0 \le t \le \in \\ 0 & t > \in \end{cases}$

And Hence $L\{f_\in(t)\} = \int_0^\infty e^{-st} f_\in(t)\,dt$

$$= \int_0^\in e^{-st} f_\in(t)\,dt + \int_\in^\infty e^{-st} f_\in(t)\,dt$$

$$= \int_0^\in e^{-st}\frac{1}{\in}dt + \int_\in^\infty e^{-st}.0\,dt$$

$$= \frac{1}{\in}\left[\frac{e^{-st}}{-s}\right]_0^\in = -\,{}^1\!/_{\in\, s}\left[e^{-s\in} - e^0\right] = \frac{1-e^{-s\in}}{s\in}$$

$\therefore L\{f_\in(t)\} = \frac{1-e^{-s\in}}{s\in}$

Now $L\{\delta(t)\} = \underset{\in\to 0}{lt}\, L\{f_\in(t)\} = \underset{\in\to 0}{lt}\, \frac{1-e^{-s\in}}{s\in}$

$\therefore L\{\delta(t)\} = 1$ *using* L-Hospital rule.

Properties of Dirac Delta Function:

1. $\int_0^\infty \delta(t)\, dt = 0$

2. $\int_0^\infty \delta(t)G(t)\, dt = G(0)$ where G(t) is some continuous function.

3. $\int_0^\infty \delta(t-a)G(t)\, dt = G(a)$ where G(t) is some continuous function.

4. $\int_0^\infty G(t)\delta^1(t-a) = -G^1(a)$

Problems

1. Prove that $L\{\delta(t-a)\} = e^{-as}$

Sol: By Translation theorem

$L\{\delta(t-a)\} = e^{-as}L\{\delta(t)\}$

$= e^{-as} \quad [\sin ce\, L\{\delta(t)\} = 1]$

2. Evaluate $\int_0^\infty \cos 2t\, \delta\left(t - \pi/3\right) dt$

Sol: By using property (3) then we get

$\int_0^\infty \delta(t-a)G(t)dt = G(a)$

Here $a = \pi/3, G(t) = \cos 2t$

$\therefore G(a) = G\left(\pi/3\right) = \cos 2\pi/3 = -1/2$

$\therefore \int_0^\infty \cos 2at\, \delta\left(t - \pi/3\right) dt = \cos 2\,\pi/3 = -\pi/2$

3. Evaluate $\int_0^\infty e^{-4t}\delta^1(t-2)\,dt$

Sol: By the 4th Property then we get

$\int_0^\infty \delta^1(t-a)G(t)\,dt = -G^1(a)$

$G(t) = e^{-4t}\ and\ a = 2$

$G^1(t) = -4.e^{-4t}$

$\therefore G^1(a) = G^1(2) = -4.e^{-8}$

$\therefore \int_0^\infty e^{-4t}\delta^1(t-2)\,dt = -G^1(a) = 4.e^{-8}$

Inverse Laplace Transforms:

If $\overline{f}(s)$ is the Laplace transforms of a function of f (t) i.e. $L\{f(t)\} = \overline{f}(s)$ then f (t) is called the inverse Laplace transform of $\overline{f}(s)$ and is written as $f(t) = L^{-1}\{\overline{f}(s)\}$

$\therefore L^{-1}$ is called the inverse L.T operator.

Table of Laplace Transforms and Inverse Laplace Transforms

S.No.	$L\{f(t)\} = \overline{f}(s)$	$L^{-1}\{\overline{f}(s)\} = f(t)$
1.	$L\{1\} = {}^{1}/_{s}$	$L^{-1}\{{}^{1}/_{s}\} = 1$
2.	$L\{e^{at}\} = \frac{1}{s-a}$	$L^{-1}\{{}^{1}/_{s} - a\} = e^{at}$
3.	$L\{e^{-at}\} = \frac{1}{s+a}$	$L^{-1}\{{}^{1}/_{s} + a\} = e^{at}$
4.	$L\{t^n\} = \frac{n!}{s^{n+1}} n\ is\ a + ve\ integer$	$L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$
5.	$L\{t^{n-1}\} = \frac{(n-1)!}{s^n}$	$L^{-1}\{{}^{1}/_{s^n}\} = \frac{t^{n-1}}{(n-1)!}, n = 1,2,3 \ldots$
6.	$L\{\sin at\} = \frac{a}{s^2+a^2}$	$L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}.\sin at$
7.	$L\{\cos at\} = \frac{s}{s^2+a^2}$	$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$
8.	$L\{\sinh at\} = \frac{a}{s^2-a^2}$	$L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$
9.	$L\{\cosh at\} = \frac{s}{s^2-a^2}$	$L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$
10.	$L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$	$L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{1}{b}.e^{at}\sin bt$
11.	$L\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$	$L^{-1}\left\{\frac{(s-a)}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$
12.	$L\{e^{at}\sinh bt\} = \frac{b}{(s-a)^2-b^2}$	$L^{-1}\left\{\frac{1}{(s-a)^2-b^2}\right\} = \frac{1}{b}.e^{at}\sinh bt$
13.	$L\{e^{at}\cosh bt\} = \frac{s-a}{(s-a)^2-b^2}$	$L^{-1}\left\{\frac{(s-a)}{(s-a)^2-b^2}\right\} = e^{at}\cosh bt$
14.	$L\{e^{-at}\sin bt\} = \frac{b}{(s+a)^2+b^2}$	$L^{-1}\left\{\frac{1}{(s+a)^2+b^2}\right\} = \frac{1}{b}.e^{-at}\sin bt$
15.	$L\{e^{-at}\cos bt\} = \frac{s+a}{(s+a)^2+b^2}$	$L^{-1}\left\{\frac{s+a}{(s+a)^2+b^2}\right\} = e^{-at}\cos bt$
16.	$L\{e^{at}f(t)\} = \overline{f}(s-a)$	$L^{-1}\{\overline{f}(s-a)\} = e^{at}L^{-1}\{\overline{f}(s)\}$
17.	$L\{e^{-at}f(t)\} = \overline{f}(s+a)$	$L^{-1}\{\overline{f}(s+a)\} = e^{-at}f(t)e^{-at}L^{-1}\{\overline{f}(s)\}$

Problems

1. Find the Inverse Laplace Transform of $\frac{s^2-3s+4}{s^3}$

Sol: $L^{-1}\left\{\frac{s^3-3s+4}{s^3}\right\} = L^{-1}\left\{1/s - 3.1/s^2 + 4/s^3\right\}$

$$= L^{-1}\left\{1/s\right\} - 3L^{-1}\left\{1/s^2\right\} + L^{-1}\left\{4/s^3\right\}$$

$$= 1 - 3t + 4.\frac{t^2}{2!} = 1 - 3t + 2t^2$$

2. Find the Inverse Laplace Transform of $\frac{s+2}{s^2-4s+13}$

Sol: $L^{-1}\left\{\frac{s+2}{s^2-4s+13}\right\} = L^{-1}\left\{\frac{s+2}{(s-2)^2+9}\right\} = L^{-1}\left\{\frac{s-2+4}{(s-2)^2+3^2}\right\}$

$$= L^{-1}\left\{\frac{s-2}{(s-2)^2+3^2}\right\} + 4.L^{-1}\left\{\frac{1}{(s-2)^2+3^2}\right\}$$

$$= e^{2t}\cos 3t + \frac{4}{3}e^{2t}\sin 3t$$

3. Find the Inverse Laplace Transform of $\frac{2s-5}{s^2-4}$

Sol: $L^{-1}\left\{\frac{2s-5}{s^2-4}\right\} = L^{-1}\left\{\frac{2s}{s^2-4} - \frac{5}{s^2-4}\right\}$

$$= 2L^{-1}\left\{\frac{s}{s^2-4}\right\} - 5L^{-1}\left\{\frac{1}{s^2-4}\right\}$$

$$= 2.\cosh 2t - 5.\frac{1}{2}\sinh 2t$$

4. Find $L^{-1}\left\{\frac{2s+1}{s(s+1)}\right\}$

Sol: $L^{-1}\left\{\frac{s+s+1}{s(s+1)}\right\} = L^{-1}\left\{\frac{1}{s+1} + \frac{1}{s}\right\}$

$$= L^{-1}\left\{\frac{1}{s+1}\right\} + L^{-1}\left\{\frac{1}{s}\right\} = e^{-t} + 1$$

5. Find $L^{-1}\left\{\frac{3s-8}{4s^2+25}\right\}$

Sol: $L^{-1}\left\{\frac{3s-8}{4s^2+25}\right\} = L^{-1}\left\{\frac{3s}{4s^2+25}\right\} - 8L^{-1}\left\{\frac{1}{4s^2+25}\right\}$

$$= \frac{3}{4}L^{-1}\left\{\frac{s}{s^2+(5/2)^2}\right\} - \frac{8}{4}L^{-1}\left\{\frac{1}{s^2+(5/2)^2}\right\}$$

$$= \frac{3}{4}.\cos\frac{5}{2}t - \frac{8}{4}.\frac{2}{5}\sin\frac{5}{2}t$$

$$=\frac{3}{4}\cos\frac{5}{2}t-\frac{4}{5}\sin\frac{5}{2}t$$

6. Find the Inverse Laplace Transform of $\frac{s}{(s+a)^2}$

Sol: $L^{-1}\left\{\frac{s}{(s+a)^2}\right\} = L^{-1}\left\{\frac{s+a-a}{(s+a)^2}\right\} = e^{-at}L^{-1}\left\{\frac{s-a}{s^2}\right\}$

$$= e^{-at}L^{-1}\left\{\frac{1}{s}-\frac{a}{s^2}\right\}$$

$$= e^{-at}\left[L^{-1}\left\{\frac{1}{s}\right\}-a.L^{-1}\left\{\frac{1}{s^2}\right\}\right]$$

$$=e^{-at}\left[1-at\right]$$

7. Find $L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}$

Sol: $Let\ \frac{3s+7}{s^2-2s-3}=\frac{A}{s+1}+\frac{B}{s-3}$

$A(s-3)+B(s+1)=3s+7$

$put\ s=3, 4B=16 \Rightarrow B=4$

$put\ s=-1, -4A=4 \Rightarrow A=-1$

$$\therefore \frac{3s+7}{s^2-2s-3}=\frac{-1}{s+1}+\frac{4}{s-3}$$

$$L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}=L^{-1}\left\{\frac{-1}{s+1}+\frac{4}{s-3}\right\}=-1L^{-1}\left\{\frac{1}{s+1}\right\}+4L^{-1}\left\{\frac{1}{s-3}\right\}$$

$$=-e^{-t}+4.e^{3t}$$

8. Find $L^{-1}\left\{\frac{s}{(s+1)^2(s^2+1)}\right\}$

Sol: $\frac{s}{(s+1)^2(s^2+1)}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{Cs+D}{s^2+1}$

$A(s+1)(s^2+1)+B(s^2+1)+(Cs+D)(s+1)^2=s$

Equating Co-efficient of s^3, A+C=0……..(1)

Equating Co-efficient of s^2, A+B+2C+D=0…….(2)

Equating Co-efficient of s, A+C+2D=1…….(3)

$put\ s=-1, 2B=-1 \Rightarrow B=-\frac{1}{2}$

Substituting (1) in (3) $2D=1 \Rightarrow D=\frac{1}{2}$

Substituting the values of B and D in (2)

i.e. $A-\frac{1}{2}+2C+\frac{1}{2}=0 \Rightarrow A+2C=0, also\ A+C=0 \Rightarrow A=0, C=0$

$$\therefore \frac{s}{(s+1)^2(s^2+1)} = \frac{\frac{-1}{2}}{(s+1)^2} + \frac{\frac{1}{2}}{s^2+1}$$

$$L^{-1}\left\{\frac{s}{(s+1)^2(s^2+1)}\right\} = \frac{1}{2}\left[L^{-1}\left\{\frac{1}{s^2+1}\right\} - L^{-1}\left\{\frac{1}{(s+1)^2}\right\}\right]$$

$$= \frac{1}{2}\left[\sin t - e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\}\right]$$

$$= \frac{1}{2}\left[\sin t - te^{-t}\right]$$

9. Find $L^{-1}\left\{\frac{s}{s^4+4a^4}\right\}$

Sol: Since $s^4+4a^4 = (s^2+2a^2)^2 - (2as)^2$

$$= (s^2+2as+2a^2)(s^2-2as+2a^2)$$

$$\therefore Let \quad \frac{s}{s^4+4a^4} = \frac{As+B}{s^2+2as+2a^2} + \frac{Cs+D}{s^2-2as+2a^2}$$

$$(As+B)(s^2-2as+2a^2) + (Cs+D)(s^2+2as+2a^2) = s$$

Solving we get $A=0, C=0, B=\frac{-1}{4a}, D=\frac{1}{4a}$

$$L\left\{\frac{s}{s^4+4a^4}\right\} = L^{-1}\left\{\frac{-\frac{1}{4a}}{s^2+2as+2a^2}\right\} + L^{-1}\left\{\frac{\frac{1}{4a}}{s^2-2as+2a^2}\right\}$$

$$= \frac{-1}{4}a.L^{-1}\left\{\frac{1}{(s+a)^2+a^2}\right\} + \frac{1}{4a}..L^{-1}\left\{\frac{1}{(s-a)^2+a^2}\right\}$$

$$= \frac{-1}{4a}.\frac{1}{a}.e^{-at}\sin at + \frac{1}{4a}.\frac{1}{a}e^{at}\sin at$$

$$= \frac{1}{4a^2}\sin at\left(e^{at}-e^{-at}\right) = \frac{1}{4a^2}.\sin at.2\sinh at = \frac{1}{2a^2}\sin at \sinh at$$

10. Find i. $L^{-1}\left\{\frac{s^2-3s+4}{s^3}\right\}$ ii. $L^{-1}\left\{\frac{3(s^2-2)^2}{2s^5}\right\}$

Sol:

i. $L^{-1}\left\{\frac{s^2-3s+4}{s^3}\right\} = L^{-1}\left\{\frac{s^2}{s^3} - \frac{3s}{s^3} + \frac{4}{s^3}\right\} = L^{-1}\left\{\frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3}\right\}$

$$= L^{-1}\left\{\frac{1}{s}\right\} - 3L^{-1}\left\{\frac{1}{s^2}\right\} + 4L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= 1 - 3t + 4\frac{t^2}{2!} = 1 - 3t + 2t^2$$

ii. $L^{-1}\left\{\frac{3(s^2-2)^2}{2s^5}\right\} = \frac{3}{2}L^{-1}\left\{\frac{(s^2-2)^2}{s^5}\right\} = \frac{3}{2}L^{-1}\left\{\frac{s^4-4s^2+4}{s^5}\right\}$

$$= \frac{3}{2}L^{-1}\left\{\frac{1}{s} - \frac{4}{s^3} + \frac{4}{s^5}\right\} + \frac{3}{2}\left\{L^{-1}\left\{\frac{1}{s}\right\} - 4L^{-1}\left\{\frac{1}{s^3}\right\} + 4L^{-1}\left\{\frac{1}{s^5}\right\}\right\}$$

$$=\frac{3}{2}\left[1-4\frac{t^2}{2!}+\frac{4t^4}{4!}\right]=\frac{3}{2}\left[1-2t^2+\frac{t^4}{6}\right]=\frac{1}{4}\left[t^4-6t^2+6\right]$$

11. Find $L^{-1}\left[\frac{s}{s^2-a^2}\right]$

Sol:

$$L^{-1}\left[\frac{s}{s^2-a^2}\right]=L^{-1}\left[\frac{2s}{2(s^2-a^2)}\right]=\frac{1}{2}L^{-1}\left[\frac{2s}{(s-a)(s+a)}\right]=\frac{1}{2}L^{-1}\left[\frac{1}{s-a}+\frac{1}{s+a}\right]$$

$$=\frac{1}{2}\left[e^{at}+e^{-at}\right]=\cosh at$$

12. Find $L^{-1}\left[\frac{4}{(s+1)(s+2)}\right]$

Sol: $L^{-1}\left[\frac{4}{(s+1)(s+2)}\right]=4L^{-1}\left[\frac{1}{(s+1)(s+2)}\right]=4\,L^{-1}\left[\frac{1}{s+1}-\frac{1}{s+2}\right]=4[e^{-t}-e^{-2t}]$

13. Find $L^{-1}\left\{\frac{1}{(s+1)^2(s^2+4)}\right\}$

Sol: $\frac{1}{(s+1)^2(s^2+4)}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{Cs+D}{s^2+4}$

$$A=\frac{2}{25},B=\frac{1}{5},C=\frac{-2}{25},D=\frac{-3}{25}$$

$$\therefore L^{-1}\left\{\frac{1}{(s+1)^2(s^2+4)}\right\}=\frac{2}{25}L^{-1}\left\{\frac{1}{s+1}\right\}+\frac{1}{5}L^{-1}\left\{\frac{1}{(s+1)^2}\right\}-\frac{2}{25}L^{-1}\left\{\frac{s}{s^2+4}\right\}-\frac{3}{25}L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$=\frac{2}{25}e^{-t}\,L^{-1}\left\{\frac{1}{s}\right\}+\frac{1}{5}e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\}-\frac{2}{25}\cos 2t-\frac{3}{25}.\frac{1}{2}\sin 2t$$

$$=\frac{2}{25}e^{-t}+\frac{1}{5}e^{-t}.t-\frac{2}{25}\cos 2t-\frac{3}{50}\sin 2t$$

14. Find $L^{-1}\left[\frac{s^2+s-2}{s(s+3)(s-2)}\right]$

Sol: $\frac{s^2+s-2}{s(s+3)(s-2)}=\frac{A}{s}+\frac{B}{s+3}+\frac{C}{s-2}$

Comparing with s^2, s, constants, we get

$$A=1/3,B=4/15,C=2/5$$

$$L^{-1}\left[\frac{s^2+s-2}{s(s+3)(s-2)}\right]=L^{-1}\left[\frac{1}{3s}+\frac{4}{15(s+3)}+\frac{2}{5(s-2)}\right]$$

$$=L^{-1}\left[\frac{1}{3s}\right]+L^{-1}\left[\frac{4}{15(s+3)}\right]+L^{-1}\left[\frac{2}{5(s-2)}\right]$$

$$=\frac{1}{3}+\frac{4}{15}e^{-3t}+\frac{2}{5}e^{2t}$$

15. **Find** $L^{-1}\left[\frac{s^2+2s-4}{(s^2+9)(s-5)}\right]$

Sol: $\frac{s^2+2s-4}{(s^2+9)(s-5)}=\frac{A}{s-5}+\frac{Bs+C}{s^2+9}$

Comparing with s^2, s, constants, we get

$$A={}^{31}\!/_{34},B={}^{3}\!/_{34},C={}^{83}\!/_{34}$$

$$L^{-1}\left[\frac{s^2+2s-4}{(s^2+9)(s-5)}\right]=L^{-1}\left[\frac{s^2+2s-4}{(s^2+9)(s-5)}\right]$$

$$=L^{-1}\left[\frac{31}{34(s-5)}\right]+L^{-1}\left[\frac{3}{34(s^2+9)}\right]+L^{-1}\left[\frac{83}{34(s^2+9)}\right]$$

$$=\frac{31}{34}e^{5t}+{}^{1}\!/_{34}\left[3\cos 3t+\frac{83}{3}\sin 3t\right]$$

First Shifting Theorem:

If $L^{-1}\left\{\overline{f}(s)\right\}=f(t),\text{then}L^{-1}\left\{\overline{f}(s-a)\right\}=e^{at}f(t)$

Proof: We have seen that $L\left\{e^{at}f(t)\right\}=\overline{f}(s-a)\ \therefore L^{-1}\left\{\overline{f}(s-a)\right\}=e^{at}f(t)=e^{at}L^{-1}\left\{\overline{f}(s)\right\}$

1. **Find** $L^{-1}\left\{\frac{1}{(s+2)^2+16}\right\}=L^{-1}\left\{\overline{f}(s+2)\right\}$

Sol: $L^{-1}\left\{\frac{1}{(s+2)^2+16}\right\}=e^{-2t}L^{-1}\left\{\frac{1}{s^2+16}\right\}$

$$=e^{-2t}.\frac{1}{4}\sin 4t=\frac{e^{-2t}\sin 4t}{4}$$

2. **Find** $L^{-1}\left\{\frac{3s-2}{s^2-4s+20}\right\}$

Sol: $L^{-1}\left\{\frac{3s-2}{s^2-4s+20}\right\}=L^{-1}\left\{\frac{3s-2}{(s-2)^2+16}\right\}=L^{-1}\left\{\frac{3(s-2)+4}{(s-2)^2+4^2}\right\}$

$$= 3L^{-1}\left\{\frac{s-2}{(s-2)^2+4^2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^2+4^2}\right\}$$

$$= 3e^{2t}L^{-1}\left\{\frac{s}{s^2+4^2}\right\} + 4e^{2t}L^{-1}\left\{\frac{1}{s^2+4^2}\right\}$$

$$= 3e^{2t}\cos 4t + 4e^{2t}\frac{1}{4}\sin 4t$$

3. **Find** $L^{-1}\left\{\frac{s+3}{s^2-10s+29}\right\}$

Sol: $L^{-1}\left\{\frac{s+3}{s^2-10s+29}\right\} = L^{-1}\left\{\frac{s+3}{(s-5)^2+2^2}\right\} = L^{-1}\left\{\frac{s-5+8}{(s-5)^2+2^2}\right\}$

$$= e^{5t}\mathrm{L}^{-1}\left\{\frac{s+8}{s^2+2^2}\right\} = e^{5t}\left\{\cos 2t + 8.\frac{1}{2}\sin 2t\right\}$$

Second shifting theorem:

If $L^{-1}\left\{\overline{f}(s)\right\} = f(t), then \quad L^{-1}\{e^{-as}\overline{f}(s)\} = G(\mathrm{t})$, where $\mathrm{G}(t) = \begin{cases} f\{t-a\} & if\ t > a \\ 0 & if\ t < a \end{cases}$

Proof: We have seen that $\mathrm{G}(t) = \begin{cases} f\{t-a\} & if\ t > a \\ 0 & if\ t < a \end{cases}$

$$then \quad L\{G(t)\} = e^{-as}.\overline{f}(s)$$

$$\therefore L^{-1}\left\{e^{-as}\overline{f}(s)\right\} = G(t)$$

1. **Evaluate (i)** $L^{-1}\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\}$ **(ii)** $L^{-1}\left\{\frac{e^{-3s}}{(s-4)^2}\right\}$

Sol: (i) $L^{-1}\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} + L^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$

Since $L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t = f(t)$, say

$\therefore$ By second Shifting theorem, we have $L^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\} = \begin{cases} \sin(t-\pi) & , if\ t > \pi \\ 0 & , if\ t < \pi \end{cases}$

or $L^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$ =sin(t-π)H(t-π)= -sint. H(t-π)

Hence $L^{-1}\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\}$ =sint-sint. H (t-π) =sint [1- H (t-π)]

Where H (t-π) is the Heaviside unit step function

(ii) Since $L^{-1}\left\{\frac{1}{(s-4)^2}\right\} = e^{4t} L^{-1}\left\{\frac{1}{s^2}\right\}$

$= e^{4t}.t = f(t)$, say

∴ By second Shifting theorem, we have $L^{-1}\left\{\frac{e^{-3s}}{(s-4)^2}\right\} = \begin{cases} e^{4(t-3)}.(t-3) & , if\ t>3 \\ 0 & , if\ t<3 \end{cases}$

or $L^{-1}\left\{\frac{e^{-3s}}{(s-4)^2}\right\} = e^{4(t-3)}.(t-3)\,\text{H(t-3)}$

Where H (t-3) is the Heaviside unit step function

Change of scale property:

If $L\{f(t)\} = \overline{f}(s)$, Then $L^{-1}\{\overline{f}(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right), a>0$

Proof: We have seen that $L\{f(t)\} = \overline{f}(s)$

$$Then \quad \overline{f}(as) = \frac{1}{a} L\left\{f\left(\frac{t}{a}\right)\right\}, a>0$$

$$\therefore L^{-1}\{\overline{f}(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right), a>0$$

1. **If** $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2} t\sin t$, **find** $L^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\}$

Sol: We have $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2} t\sin t$,

Writing as for s,

$L^{-1}\left\{\frac{as}{(a^2 s^2+1)^2}\right\} = \frac{1}{2}.\frac{1}{a}.\frac{t}{a}\sin\frac{t}{a} = \frac{t}{2a^2}.\sin\frac{t}{a}$, by change of scale property.

Putting a=2, we get

$L^{-1}\left\{\frac{2s}{(4s^2+1)^2}\right\} = \frac{t}{8}\sin\frac{t}{2}$ **or** $L^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\} = \frac{1}{2}\sin\frac{t}{2}$

Inverse Laplace Transform of derivatives:

Theorem: $L^{-1}\{\overline{f}(s)\} = f(t)$, then $L^{-1}\{\overline{f}^n(s)\} = (-1)^n t^n f(t)$ where $\overline{f}^n(s) = \frac{d^n}{ds^n}\left[\overline{f}(s)\right]$

Proof: We have seen that $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} \overline{f}(s)$

$$\therefore L^{-1}\left\{\overline{f}^{n}(s)\right\} = (-1)^{n} t^{n} f(t)$$

1. Find $L^{-1}\left\{\log\frac{s+1}{s-1}\right\}$

Sol: Let $L^{-1}\left\{\log\frac{s+1}{s-1}\right\} = f(t)$

$$L\{f(t)\} = \log\frac{s+1}{s-1}$$

$$L\{tf(t)\} = \frac{-d}{ds}\left\{\log\frac{s+1}{s-1}\right\}$$

$$L\{tf(t)\} = \frac{-1}{s+1} + \frac{1}{s-1}$$

$$tf(t) = L^{-1}\left\{\frac{-1}{s+1} + \frac{1}{s-1}\right\}$$

$$tf(t) = -1.\mathrm{L}^{-1}\left\{\frac{1}{s+1}\right\} + L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= e^{-t} + e^{t}$$

$$t\,f(t) = 2\sinh t \Rightarrow f(t) = \frac{2\sinh t}{t}$$

$$\therefore L^{-1}\left\{\log\frac{s+1}{s-1}\right\} = \frac{2\sinh t}{t}$$

Note: $L^{-1}\left\{\log\frac{1+s}{s}\right\} = \frac{1-e^{-t}}{t}$

2. Find $L^{-1}\left\{\cot^{-1}(s)\right\}$

Sol: Let $L^{-1}\left\{\cot^{-1}(s)\right\} = f(t)$

$$L\{f(t)\} = \cot^{-1}(s)$$

$$L\{tf(t)\} = \frac{-d}{ds}[\cot^{-1}(s) = -\left[\frac{-1}{1+s^{2}}\right] = \frac{1}{1+s^{2}}$$

$$tf(t) = L^{-1}\left\{\frac{1}{s^{2}+1}\right\} = \sin t$$

$$f(t) = \frac{\sin t}{t}$$

$$\therefore L^{-1}\left\{\cot^{-1}(s)\right\} = \frac{1}{\mathrm{t}}\sin t$$

Inverse Laplace Transform of integrals:

Theorem: $L^{-1}\left\{\overline{f}(s)\right\} = f(t)$, then $L^{-1}\left\{\int_s^\infty \overline{f}(s)ds\right\} = \frac{f(t)}{t}$

Proof: we have seen that $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \overline{f}(s)ds$

$$\therefore L^{-1}\left\{\int_s^\infty \overline{f}(s)ds\right\} = \frac{f(t)}{t}$$

1. Find $L^{-1}\left\{\frac{s+1}{(s^2+2s+2)^2}\right\}$

Sol: Let $\overline{f}(s) = \frac{s+1}{(s^2+2s+2)^2}$

Then $L^{-1}\left\{\overline{f}(s)\right\} = L^{-1}\left\{\int_s^\infty \frac{s+1}{(s^2+2s+2)^2}ds\right\}$

$$= L^{-1}\left\{\frac{s+1}{[(s+1)^2+1]^2}\right\}$$

$$= e^{-t}L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}, \text{ by First Shifting Theorem}$$

$$= e^{-t}\frac{t}{2}\sin t = \frac{t}{2}e^{-t}\sin t \because L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t}{2a}\sin at$$

Multiplication by power of's':

Theorem: $L^{-1}\left\{\overline{f}(s)\right\} = f(t)$, and f (0),then $L^{-1}\left\{s\overline{f}(s)\right\} = f^1(t)$

Proof: we have seen that $L\left\{f^1(t)\right\} = s\overline{f}(s) - f(0)$

$\therefore L\left\{f^1(t)\right\} = s\overline{f}(s)$ $[\because f(0)=0]$ or

$L^{-1}\left\{s\overline{f}(s)\right\} = f^1(t)$

Note: $L^{-1}\left\{s^n\overline{f}(s)\right\} = f^n(t), \text{if } f^n(0) = 0 \text{ for } n = 1,2,3.........n-1$

1. Find (i) $L^{-1}\left\{\frac{s}{(s+2)^2}\right\}$ **(ii)** $L^{-1}\left\{\frac{s}{(s+3)^2}\right\}$

Sol: Let $\overline{f}(s) = \frac{1}{(s+2)^2}$ **Then**

$$L^{-1}\{\overline{f}(s)\} = L^{-1}\left\{\frac{1}{(s+2)^2}\right\} = e^{-2t}L^{-1}\left\{\frac{1}{s^2}\right\} = e^{-2t}.t = f(t),$$

Clearly f (0) =0

Thus $L^{-1}\left\{\frac{s}{(s+2)^2}\right\} = L^{-1}\left\{s.\frac{s}{(s+2)^2}\right\} = L^{-1}\{s.\overline{f}(s)\} = f^1(t)$

$$= \frac{d}{dt}(te^{-2t}) = \text{t}(-2e^{-2t}) + e^{-2t}.1 = e^{-2t}(1-2\text{t})$$

Note: in the above problem put 2=3, then $L^{-1}\left\{\frac{s}{(s+3)^2}\right\} = e^{-3t}(1-3\text{t})$

Division by S:

Theorem: If $L^{-1}\{\overline{f}(s)\} = f(t)$, Then $L^{-1}\left\{\frac{\overline{f}(s)}{s}\right\} = \int_0^t f(u)\,du$

Proof: We have seen that $L\left\{\int_0^t f(u)\,du\right\} = \frac{\overline{f}(s)}{s}$

$$\therefore L^{-1}\left\{\frac{\overline{f}(s)}{s}\right\} = \int_0^t f(u)\,du$$

Note: If $L^{-1}\{\overline{f}(s)\} = f(t)$, then $L^{-1}\left\{\frac{\overline{f}(s)}{s^2}\right\} = \int_0^t\int_0^t f(u)\,du.du$

1. Find the inverse Laplace Transform of $\frac{1}{s^2(s^2+a^2)}$

Sol: Since $L^{-1}\left[\frac{1}{(s^2+a^2)}\right] = \frac{1}{a}\text{sinat}$, we have

$$L^{-1}\left[\frac{1}{\text{s}(s^2+a^2)}\right] = \int_0^t \frac{1}{a}\sin at\,dt$$

$$= \frac{1}{a}\left(\frac{-\cos at}{a}\right)_0^t = -\frac{1}{a^2}(\text{cosat}-1) = \frac{1}{a^2}(1-\text{cosat})$$

Then $L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right] = \int_0^t \frac{1}{a^2}(1-\cos at)\,dtdt$

$$= \frac{1}{a^2}\left(t - \frac{\sin at}{a}\right)_0^t = \frac{1}{a^2}\left(t - \frac{\sin at}{a}\right)$$

$$\therefore L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right] = \frac{1}{a^2}\left(t - \frac{\sin at}{a}\right)$$

Convolution Definition:

If f (t) and g (t) are two functions defined for $t \geq 0$ then the convolution of f (t) and g (t) is defined as $f(t)*g(t) = \int_0^t f(u)g(t-u)\,du$

$f(t)*g(t)$ *can also be written as* $(f*g)(t)$

Properties:

The convolution operation * has the following properties

1. Commutative i.e. $(f*g)(t) = (g*f)(t)$

2. Associative $[f*(g*h)](t) = [(f*g)*h](t)$

3. Distributive $[f*(g+h)](t) = (f*g)(t) + (f*h)(t)$ *for* $t \geq 0$

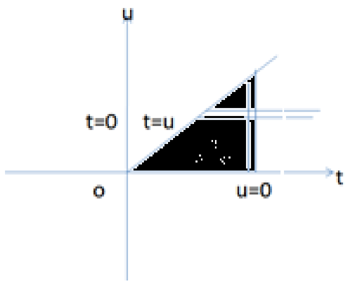
Convolution Theorem: If $f(t)$ **and** $g(t)$ **are functions defined for** $t \geq 0$ **then**

$$L\{f(t)*g(t)\} = L\{f(t)\}.L\{g(t)\} = \bar{f}(s).\bar{g}(s)$$

i.e., The L.T of convolution of f(t) and g(t) is equal to the product of the L.T of f(t) and g(t)

Proof: WKT $L\{\phi(t)\} = \int_0^\infty e^{-st}\left\{\int_0^t f(u)\,g(t-u)\,du\right\}dt$

$$= \int_0^\infty \int_0^t e^{-st} f(u)\,g(t-u)\,du\,dt$$



The double integral is considered within the region enclosed by the line u=0 and u=t

On changing the order of integration, we get

$$L\{\phi(t)\} = \int_0^\infty \int_u^\infty e^{-st} f(u)\,g(t-u)\,dt\,du$$

$$= \int_0^\infty e^{-su} f(u)\left\{\int_u^\infty e^{-s(t-u)} g(t-u)\,dt\right\}du$$

$$= \int_0^\infty e^{-su} f(u)\left\{\int_0^\infty e^{-sv} g(v)\,dv\right\}du \quad put\ t-u=v$$

$$= \int_0^\infty e^{-su} f(u)\{\bar{g}(s)\}du = \bar{g}(s)\int_0^\infty e^{-su} f(u)\,du = \bar{g}(s).\bar{f}(s)$$

$$L\{f(t)*g(t)\} = L\{f(t)\}.L\{g(t)\} = \bar{f}(s).\bar{g}(s)$$

Problems:

1. Using the convolution theorem find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

Sol: $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}=L^{-1}\left\{\frac{s}{s^2+a^2}.\frac{1}{s^2+a^2}\right\}$

$Let\ \overline{f}(s)=\frac{s}{s^2+a^2}\ and\ \overline{g}(s)=\frac{1}{s^2+a^2}$

So that $L^{-1}\{\overline{f}(s)\}=L^{-1}\left\{\frac{s}{s^2+a^2}\right\}=\cos at=f(t)-say$

$L^{-1}\{\overline{g}(s)\}=L^{-1}\left\{\frac{s}{s^2+a^2}\right\}=\frac{1}{a}\sin at=g(t)\to say$

$\therefore$ By convolution theorem, we have

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}=\int_0^t \cos au.\frac{1}{a}.\sin a(t-u)du$$

$$=\frac{1}{2a}\int_0^t[\sin(\text{au}+at-au)-\sin(au-at+au)]du$$

$$=\frac{1}{2a}\int_0^t[\sin at-\sin(2au-at)]du$$

$$=\frac{1}{2a}\left[\sin at.u+\frac{1}{2a}.\cos(2au-at)\right]_0^t$$

$$=\frac{1}{2a}\left[t\sin at+\frac{1}{2a}\cos(2at-at)-\frac{1}{2a}\cos(-at)\right]$$

$$=\frac{1}{2a}\left[t\sin at+\frac{1}{2a}\cos at-\frac{1}{2a}\cos at\right]$$

$$=\frac{t}{2a}\sin at$$

2. Use convolution theorem to evaluate $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

Sol: $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}=L^{-1}\left\{\frac{s}{s^2+a^2}.\frac{s}{s^2+b^2}\right\}$

$Let\ \overline{f}(s)=\frac{s}{s^2+a^2}\ and\ \overline{g}(s)=\frac{s}{s^2+b^2}$

So that $L^{-1}\{\overline{f}(s)\}=L^{-1}\left\{\frac{s}{s^2+a^2}\right\}=\cos at=f(t)\to say$

$$L^{-1}\left\{\bar{g}(s)\right\} = L^{-1}\left\{\frac{s}{(s^2+b^2)}\right\} = \cos bt = g(t) \rightarrow say$$

$\therefore$ By convolution theorem, we have

$$L^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right\} = \int_0^t \cos au.\cos b(t-u)du$$

$$= \frac{1}{2}\int_0^t \left[\cos(au-bu+bt)+\cos(au+bu-bt)\right]du$$

$$= \frac{1}{2}\left[\frac{\sin(au-bu+bt)}{a-b}+\frac{\sin(au+bu-bt)}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin at-\sin bt}{a-b}+\frac{\sin at+\sin bt}{a+b}\right] = \frac{a\sin at-b\sin bt}{a^2-b^2}$$

3. **Use convolution theorem to evaluate** $L^{-1}\left\{\frac{1}{s(s^2+4)^2}\right\}$

Sol: $L^{-1}\left\{\frac{1}{s(s^2+4)^2}\right\} = L^{-1}\left\{\frac{1}{s^2}\cdot\frac{s}{(s^2+4)^2}\right\}$

$$Let\ \bar{f}(s) = \frac{1}{s^2}\ and\ \bar{g}(s) = \frac{s}{\left(s^2+4\right)^2}$$

So that $L^{-1}\left\{\bar{g}(s)\right\} = L^{-1}\left\{\frac{1}{s^2}\right\} = t = g(t) \rightarrow say$

$$L^{-1}\left\{\bar{f}(s)\right\} = L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{t.\sin 2t}{4} = f(t) - say\left[\therefore L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t\sin 2t}{2a}\right]$$

$$\therefore L^{-1}\left\{\frac{1}{s^2}\cdot\frac{s}{(s^2+4)^2}\right\} = \int_0^t \frac{u}{4}\sin 2u(t-u)du$$

$$= {t}/{4}\int_0^t u\sin 2u du - \frac{1}{4}\int_0^t u^2\sin 2u du$$

$$= {t}/{4}\left(-\frac{u}{2}\cos 2u+\frac{1}{4}\sin 2u\right)_0^t$$

$$= -\frac{1}{4}\left[\frac{-u^2}{2}\cos 2u+\frac{u}{2}\sin 2u+\frac{1}{4}\cos 2u\right]_0^t$$

$$= \frac{1}{16}[1-t\sin 2t-\cos 2t]$$

4. Find $L^{-1}\left[\frac{1}{(s-2)(s^2+1)}\right]$

Sol: $L^{-1}\left[\frac{1}{(s-2)(s^2+1)}\right]=L^{-1}\left[\frac{1}{s-2}.\frac{1}{s^2+1}\right]$

$Let\ \overline{f}(s)=\frac{1}{s-2}\ and\ \overline{g}(s)=\frac{1}{s^2+1}$

So that $L^{-1}\left\{\overline{f}(s)\right\}=L^{-1}\left\{\frac{1}{s-2}\right\}=e^{2t}=f(t)\to say$

$L^{-1}\left\{\overline{g}(s)\right\}=L^{-1}\left\{\frac{1}{s^2+1}\right\}=\sin t=g(t)\to say$

$\therefore L^{-1}\left\{\frac{1}{s-2}.\frac{1}{s^2+1}\right\}=\int_0^t f(u).\mathrm{g}(\mathrm{t}-u)\,\mathrm{du}$ (By Convolution theorem)

$$=\int_0^t e^{2u}\sin(\mathrm{t}-\mathrm{u})\,\mathrm{du}\ \ (\mathrm{or})\ \int_0^t \sin\mathrm{u}.e^{2(t-u)}\,\mathrm{du}$$

$$=e^{2t}\int_0^t \sin u e^{-2u}du$$

$$=e^{2t}\left[\frac{e^{-2u}}{2^2+1}\left[-2\sin u-\cos u\right]\right]_0^t$$

$$=e^{2t}\left[\frac{1}{5}e^{-2t}\left(-2\sin t-\cos t\right)-\frac{1}{5}(-1)\right]$$

$$=\frac{1}{5}\left(e^{2t}-2\sin t-\cos t\right)$$

5. Find $L^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\}$

Sol: $L^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\}=L^{-1}\left\{\frac{1}{s+1}.\frac{1}{s-2}\right\}$

$Let\ \overline{f}(s)=\frac{1}{s+1}\ and\ \overline{g}(s)=\frac{1}{s-2}$

So that $L^{-1}\left\{\overline{f}(s)\right\}=L^{-1}\left\{\frac{1}{s+1}\right\}=e^{-t}=f(t)\to say$

$L^{-1}\left\{\overline{g}(s)\right\}=L^{-1}\left\{\frac{1}{s-2}\right\}=e^{2t}=g(t)\to say$

$\therefore$ By using convolution theorem, we have

$$L^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\}=\int_0^t e^{-u}e^{2(t-u)}du$$

$$=\int_0^t e^{2t}e^{-3u}du=e^{2t}\int_0^t e^{-3u}du=e^{2t}\left[\frac{e^{-3u}}{-3}\right]_0^t=\frac{1}{3}\left[e^{2t}-e^{-t}\right]$$

6. **Find** $L^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\}$

Sol: $L^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\}=L^{-1}\left\{\frac{1}{s^2}.\frac{1}{s^2-a^2}\right\}$

$Let\,\overline{f}(s)=\frac{1}{s^2}\ and\ \overline{g}(s)=\frac{1}{s^2-a^2}$

So that $L^{-1}\left\{\overline{f}(s)\right\}=L^{-1}\left\{\frac{1}{s^2}\right\}=t=f(t)-say$

$L^{-1}\left\{\overline{g}(s)\right\}=L^{-1}\left\{\frac{1}{s^2-a^2}\right\}=\frac{1}{a}\sinh at=g(t)-say$

By using convolution theorem, we have

$$L^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\}=\int_0^t u.\frac{1}{a}\sinh a(t-u)du$$

$$=\frac{1}{a}\int_0^t u\sinh(at-au)du$$

$$=\frac{1}{a}\left[\frac{-u}{a}\cosh\left(at-au\right)-\frac{\sin\left(at-au\right)}{a^2}\right]_0^t$$

$$=\frac{1}{a}\left[\frac{-t}{a}\cosh(at-at)-0-\frac{1}{a^2}[0-\sinh at]\right]$$

$$=\frac{1}{a}\left[\frac{-t}{a}+\frac{1}{a^2}\sinh at\right]$$

$$=\frac{1}{a^3}\left[-at+\sinh at\right]$$

3. **Using Convolution theorem, evaluate** $L^{-1}\{\frac{s}{(s+2)(s^2+9)}\}$

Sol: $L^{-1}\left\{\frac{1}{s+2}.\frac{s}{s^2+9}\right\}=L^{-1}\{\frac{1}{s+2}.\frac{s}{s^2+3^2}\}$=$L^{-1}\{\bar{f}(s).\bar{g}\,(s)\}$

$\bar{f}(s)$=$\frac{1}{s+2}=L\{f(t)\}\Rightarrow f(t)=L^{-1}\left\{\frac{1}{s+2}\right\}=e^{-2t}$----------------- (1)

$\bar{g}(s)$=$\frac{s}{s^2+3^2}=L\{g(t)\}\Rightarrow g(t)=L^{-1}\left\{\frac{s}{s^2+3^2}\right\}=cos3t$----------------- (2)

By Convolution theorem we have

$$L^{-1}\{\bar{f}(s).\bar{g}\,(s)\} = f(t) * g(t)$$

Where $f(t) * g(t) = \int_0^t g(u)f(t-u)du$

$$\therefore \qquad L^{-1}\left\{\frac{1}{s+2}\cdot\frac{s}{s^2+9}\right\} = \int_0^t e^{-2(t-u)}cos3udu$$

$$= e^{-2t}\int_0^t e^{2u}cos3udu$$

$$= e^{-2t}\cdot\frac{1}{2^2+3^2}[2cos3u - 3sin3u]_0^t$$

$$= \frac{e^{-2t}}{13}[2cos3t - 2 - 3sin3t]$$

$$= \frac{1}{13}[e^{-2t}(2cos3t - 3sin3t)] - \frac{2e^{-2t}}{13}$$

Application of L.T to ordinary differential equations:

(Solutions of ordinary DE with constant coefficient):

1. **Step1:** Take the Laplace Transform on both the sides of the DE and then by using the formula
$L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-1} f^1(0) - s^{n-2} f^2(0) - ....f^{n-1}(0)$ and apply given initial conditions. This gives an algebraic equation.
2. **Step2:** replace f (0), $f^1(0)$, $f^2(0)$,……… $f^{n-1}(0)$ with the given initial conditions.
Where $f^1(0) = s\overline{f}(0) - f(0)$
$f^2(0) = s^2\overline{f}(s) - sf(0) - f^1(0)$, and so on
3. **Step3:** solve the algebraic equation to get derivatives in terms of s.
4. **Step4:** take the inverse Laplace transform on both sides this gives f as a function of t which gives the solution of the given DE

Problems:

1. Solve $y^{111} + 2y^{11} - y^1 - 2y = 0$ using Laplace Transformation given that $y(0) = y^1(0) = 0$ and $y^{11}(0) = 6$

Sol: Given that $y^{111} + 2y^{11} - y^1 - 2y = 0$

Taking the Laplace transform on both sides, we get

$$\mathrm{L}\left\{y^{111}(t)\right\} + 2\mathrm{L}\left\{y^{11}(t)\right\} - \mathrm{L}\left\{y^1\right\} - 2\mathrm{L}\left\{y\right\} = 0$$

$$\Rightarrow s^3\mathrm{L}\left\{y(t)\right\} - s^2y(0) - sy^1(0) - y^{11}(0) + 2\left\{s^2\mathrm{L}\left\{y(t)\right\} - sy(0) - y^1(0)\right\} -$$

$$\left\{s\mathrm{L}\left\{y(t)\right\} - y(0)\right\} - 2\mathrm{L}\left\{y(t)\right\} = 0$$

$$\Rightarrow \left\{s^3+2s^2-s-2\right\} L\left\{y(t)\right\} = s^2 y(0) + sy^1(0) + y^{11}(0) + 2sy(0) + 2y^1(0) - y(0)$$

$$= 0+0+6+2.0+2.0-0$$

$$\Rightarrow \left\{s^3+2s^2-s-2\right\} L\left\{y(t)\right\} = 6$$

$$L\left\{y(t)\right\} = \frac{6}{s^3+2s^2-s-2} = \frac{6}{(s-1)(s+1)(s+2)}$$

$$= \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$\Rightarrow A(s+1)(s+2) + B(s-1)(s+2) + C(s-1)(s+1) = 6$$

$$\Rightarrow A(s^2+3s+2) + B(s^2-s-2) + C(s^2-1) = 6$$

Comparing both sides s^2,s,constants,we have

$$\Rightarrow A+B+C=0, 3A-B=0, 2A-2B-C=6$$

$$A+B+C=0$$
$$2A-2B-C=6$$

$$3A-B=6$$
$$3A+B=0$$

$$6A=6 \Rightarrow A=1$$
$$3A+B=0 \Rightarrow B=-3A \Rightarrow B=-3$$

$$\therefore A+B+C=0 \Rightarrow C=-A-B=-1+3=2$$

$$\therefore L\left\{y(t)\right\} = \frac{1}{s-1} - \frac{3}{s+1} + \frac{2}{s+2}$$

$$y(t) = L^{-1}\left\{\frac{1}{s-1}\right\} - 3.L^{-1}\left\{\frac{1}{s+1}\right\} + 2.L^{-1}\left\{\frac{1}{s+2}\right\} = e^t - 3e^{-t} + 2.e^{-2t}$$

Which is the required solution

2. **Solve** $y^{11} - 3y^1 + 2y = 4t + e^{3t}$ **using Laplace Transformation given that** $y(0) = 1 \, and \, y^1(0) = -1$

Sol: Given that $y^{11} - 3y^1 + 2y = 4t + e^{3t}$

Taking the Laplace transform on both sides, we get

$$L\left\{y^{11}(t)\right\} - 3L\left\{y^1(t)\right\} + 2L\left\{y(t)\right\} = 4L\left\{t\right\} + L\left\{e^{3t}\right\}$$

$$\Rightarrow s^2 L\left\{y(t)\right\} - sy(0) - y^1(0) - 3\left[sL\left\{y(t)\right\} - y(0)\right] + 2L\left\{y(t)\right\} = \frac{4}{s^2} + \frac{1}{s-3}$$

$$\Rightarrow (s^2-3s+2)L\{y(t)\} = \frac{4}{s^2}+\frac{1}{s-3}+s-4$$

$$\Rightarrow (s^2-3s+2)L\{y(t)\} = \frac{4s-12+s^4+s^2-3s^3-4s^3+12s^2}{s^2(s-3)}$$

$$\Rightarrow L\{y(t)\} = \frac{s^4-7s^3+13s^2+4s-12}{s^2(s-3)(s^2-3s+2)}$$

$$\Rightarrow L\{y(t)\} = \frac{s^4-7s^3+13s^2+4s-12}{s^2(s-3)(s-1)(s-2)}$$

$$\Rightarrow \frac{s^4-7s^3+13s^2+4s-12}{s^2(s-3)(s-1)(s-2)} = \frac{As+B}{s^2}+\frac{C}{s-3}+\frac{D}{s-1}+\frac{E}{s-2}$$

$$= \frac{(As+B)(s-1)(s-2)(s-3)+C(s^2)(s-1)(s-2)+D(s^2)(s-2)(s-3)+E(s^2)(s-1)(s-3)}{s^2(s-3)(s-1)(s-2)}$$

$$\Rightarrow s^4-7s^3+13s^2+4s-12 = (As+B)(s^3-6s^2+11s-6)+$$
$$C(s^2)(s^2-3s+2)+D(s^2)(s^2-5s+6)+E.s^2(s^2-4s+3)$$

Comparing both sides s^4,s^3,we have

$A+C+D+E=1$......................(1)

$-6A+B-3C-5D-4E=-7$......................(2)

$put\ s=1, 2D=-1 \Rightarrow D=\frac{-1}{2}$

$put\ s=2, -4E=8 \Rightarrow E=-2$

$put\ \ s=3, 18C=9 \Rightarrow C=\frac{1}{2}$

$from\,eq.(1)\ A=1-\frac{1}{2}+\frac{1}{2}+2 \Rightarrow A=3$

from eq.(2) B= -7+18+$\frac{3}{2}-\frac{5}{2}-8=3-1=2$

$$y(t)=L^{-1}\left\{\frac{3}{s}+\frac{2}{s^2}+\frac{1}{2(s-3)}-\frac{1}{2(s-1)}-\frac{2}{s-2}\right\}$$

$$y(t)=3+2t+\frac{1}{2}e^{3t}-\frac{1}{2}e^{t}-2.e^{2t}$$

3. Using Laplace Transform Solve $\frac{d^2y}{dt^2}+2\frac{dy}{dt}-3y=\sin t,$ **given that** $y=\frac{dy}{dt}=0$ **when t=0**

Sol: Given equation is $\frac{d^2y}{dt^2}+2\frac{dy}{dt}-3y=\sin t.$

$$L\{y^{11}(t)\}+2L\{y^{1}(t)\}-3L\{y(t)\}=L\{\sin t\}$$

$$s^2L\{y(t)\}-sy(0)-y^1(0)+2\left[sL\{y(t)\}-y(0)\right]-3.L\{y(t)\}=\frac{1}{s^2+1}$$

$$\Rightarrow(s^2+2s-3)L\{y(t)\}=\frac{1}{s^2+1}$$

$$\Rightarrow L\{y(t)\}=\left(\frac{1}{(s^2+1)(s^2+2s-3)}\right)$$

$$\Rightarrow y(t)=L^{-1}\left(\frac{1}{(s-1)(s+3)(s^2+1)}\right)$$

Now consider

$$\frac{1}{(s-1)(s+3)(s^2+1)}=\frac{A}{s-1}+\frac{B}{s+3}+\frac{Cs+D}{s^2+1}$$

$$A(s+3)(s^2+1)+B(s-1)(s^2+1)+(Cs+D)(s-1)(s+3)=1$$

Comparing both sides s^3,we have

$$put\ s=1, 8A=1\Rightarrow A=\frac{1}{8}$$

$$put\ s=-3, -40B=1\Rightarrow B=\frac{-1}{40}$$

$$A+B+C=0\Rightarrow C=0-\frac{1}{8}+\frac{1}{40}$$

$$C=\frac{-5+1}{40}=\frac{-4}{40}=\frac{-1}{10}$$

$$3A-B+2C+D=0\Rightarrow D=-\frac{3}{8}-\frac{1}{40}+\frac{1}{5}$$

$$D=\frac{-15-1+8}{40}=\frac{-8}{40}=\frac{-1}{5}$$

$$\therefore y(t)=L^{-1}\left\{\frac{\frac{1}{8}}{s-1}+\frac{\frac{-1}{40}}{s+3}+\frac{\frac{-1}{10}s-\frac{1}{5}}{s^2+1}\right\}$$

$$=\frac{1}{8}L^{-1}\left\{\frac{1}{s-1}\right\}-\frac{1}{40}L^{-1}\left\{\frac{1}{s+3}\right\}-\frac{1}{10}L^{-1}\left\{\frac{s}{s^2+1}\right\}-\frac{1}{5}L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\therefore y(t)=\frac{1}{8}e^{t}-\frac{1}{40}e^{-3t}-\frac{1}{10}\cos t-\frac{1}{5}\sin t$$

4. Solve $\frac{dx}{dt}+x=\sin\omega t, x(0)=2$

Sol: Given equation is $\frac{dx}{dt}+x=\sin\omega t$

$$L\{x^1(t)\}+L\{x(t)\}=L\{\sin\omega t\}$$

$$\Rightarrow s.L\{x(t)\}-x(0)+L\{x(t)\}=\frac{\omega}{s^2+\omega^2}$$

$$\Rightarrow s.L\{x(t)\}-2+L\{x(t)\}=\frac{\omega}{s^2+\omega^2}$$

$$\Rightarrow (s+1)L\{x(t)\}=\frac{\omega}{s^2+\omega^2}+2$$

$$\Rightarrow x(t)=L^{-1}\left\{\frac{\omega}{(s+1)(s^2+\omega^2)}+\frac{2}{s+1}\right\}$$

$$=2L^{-1}\left\{\frac{1}{s+1}\right\}+L^{-1}\left\{\frac{\omega}{(s+1)(s^2+\omega^2)}\right\}$$ (By using partial fractions)

$$=2e^{-t}+L^{-1}\left\{\frac{\frac{\omega}{\omega^2+1}}{s+1}-\frac{\frac{s\omega}{1+\omega^2}}{s^2+\omega^2}+\frac{\frac{\omega}{1+\omega^2}}{s^2+\omega^2}\right\}$$

$$=2e^{-t}+\frac{\omega}{\omega^2+1}e^{-t}-\frac{\omega}{1+\omega^2}\cos\omega t+\frac{\omega}{1+\omega^2}.\frac{1}{\omega}\sin\omega t$$

5. Solve $(D^2+n^2)x=a\sin(nt+\alpha)$ **given that x=Dx=0, when t=0**

Sol: Given equation is $(D^2+n^2)x=a\sin(nt+\alpha)$

$$x^{11}(t)+n^2x(t)=a\sin(nt+\alpha)$$

$$L\{x^{11}(t)\}+n^2L\{x(t)\}=L\{a\sin nt\cos\alpha+a\cos nt\sin\alpha\}$$

$$\Rightarrow s^2L\{x(t)\}-sx(0)-x^1(0)+n^2L\{x(t)\}=a\cos\alpha L\{\sin nt\}+a\sin\alpha L\{\cos nt\}$$

$$\Rightarrow (s^2+n^2)L\{x(t)\}=a\cos\alpha\frac{n}{s^2+n^2}+a\sin\alpha.\frac{s}{s^2+n^2}$$

$$\Rightarrow L\{x(t)\}=a\cos\alpha\frac{n}{(s^2+n^2)^2}+a\sin\alpha\frac{s}{(s^2+n^2)^2}$$

(By using convolution theorem I –part, partial fraction in II-part)

$$= na\cos\alpha\int_0^t \frac{1}{n}.\sin nx.\frac{1}{n}\sin n(t-x)\,dx - \frac{a\sin\alpha}{2}L^{-1}\left\{\frac{d}{ds}\frac{1}{(s^2+n^2)}\right\}$$

$$= \frac{a\cos\alpha}{2n}\int_0^t \{\cos(nt-2nx)-\cos nt\}\,dx + \frac{a\sin\alpha}{2}t\frac{1}{n}\sin nt$$

$$= \frac{a\cos\alpha}{2n}\left[\int_0^t \{\cos n(t-2x)-\cos nt\}\,dx + \frac{a}{2n}\sin\alpha t\sin nt\right]$$

$$= \frac{a\cos\alpha}{2n}\left[\frac{-1}{2n}.\sin n(t-2x) - x\cos nt\right]_0^t + \frac{at\sin\alpha}{2n}\sin nt$$

$$= \frac{a\cos\alpha}{2n}\left[\frac{\sin nt}{2n} - t\cos nt\right] + \frac{at\sin\alpha}{2n}\sin nt$$

$$= \frac{a\cos\alpha\sin nt}{2n^2} - \frac{at}{2n}[\cos\alpha\cos nt - \sin\alpha\sin nt]$$

$$= \frac{a\cos\alpha\sin nt}{2n^2} - \frac{at}{2n}\cos(\alpha+nt)$$

6. Solve $y^{11} - 4y^1 + 3y = e^{-t}$ using L.T given that y (0) = y^1 (0) = 1.

Sol: Given equation is $y^{11} - 4y^1 + 3y = e^{-t}$

Applying L.T on both sides we get $L(y^{11}) - 4L(y^1) + 3L(y) = L(e^{-t})$

$\Rightarrow \{s^2L[y] - s\,y(0) - y^1(0)\} - 4\{s\,L[y] - y(0)\} + 3L\{y\} = \frac{1}{s+1}$

$\Rightarrow (s^2 + 4s + 3)\,L\{y\} - s - 1 - 4 = \frac{1}{s+1}$

$\Rightarrow (s^2 + 4s + 3)\,L\{y\} = \frac{1}{s+1} + s + 5$

$\Rightarrow (s^2 + 4s + 3)\,L\{y\} = \frac{1}{s+1} + s + 5$

$$L\{y\} = \frac{1}{(s+1)(s^2+4s+3)} + \frac{s+5}{(s^2+4s+3)}$$

$$y = L^{-1}\left[\frac{1}{(s+1)(s^2+4s+3)}\right] + L^{-1}\left[\frac{s+5}{(s^2+4s+3)}\right]$$

Let us consider

$$L^{-1}\left[\frac{1}{(s+1)(s^2+4s+3)}\right] = L^{-1}\left[\frac{1}{(s+1)^2(s+3)}\right]$$

$$\frac{1}{(s+1)(s^2+4s+3)} = \frac{1}{(s+1)^2(s+3)}$$

$$= \frac{A}{s+1} + \frac{B}{(S+1)^2} + \frac{C}{S+3}$$

$$= \frac{(-\frac{1}{4})}{s+1} + \frac{(\frac{1}{2})}{(S+1)^2} + \frac{(\frac{1}{4})}{S+3}$$

$$= L^{-1}\left[\frac{(-\frac{1}{4})}{s+1} + \frac{(\frac{1}{2})}{(S+1)^2} + \frac{(\frac{1}{4})}{S+3}\right]$$

$=L^{-1}[\frac{(-\frac{1}{4})}{s+1}+\frac{(\frac{1}{2})}{(S+1)^2}+\frac{(\frac{1}{4})}{S+3}]$

$=-\frac{1}{4}L^{-1}[\frac{1}{s+1}]+\frac{1}{2}L^{-1}[\frac{1}{(S+1)^2}]+\frac{1}{4}L^{-1}[\frac{1}{S+3}]$

$$L^{-1}\left[\frac{1}{(s+1)(s^2+4s+3)}\right]=-\frac{1}{4}e^{-t}+\frac{1}{2}te^{-t}+\frac{1}{4}e^{-3t}---\rightarrow(1)$$

$L^{-1}\left[\frac{s+5}{(s^2+4s+3)}\right]=L^{-1}[\frac{s+2}{((s+2)^2-1)}]+L^{-1}[\frac{3}{((s+2)^2-1)}]$

$=\boldsymbol{e^{-2t}}L^{-1}[\frac{s}{(s^2-1)}]+L^{-1}+\boldsymbol{3e^{-2t}}L^{-1}[\frac{1}{(s^2-1)}]$

$L^{-1}\left[\frac{s+5}{(s^2+4s+3)}\right]=\boldsymbol{cost+3e^{-2t}sint}---\rightarrow(2)$

From (1) & (2)

$\therefore y=-\frac{1}{4}e^{-t}+\frac{1}{2}te^{-t}+\frac{1}{4}e^{-3t}+\boldsymbol{e^{-2t}cost+3e^{-2t}sint}$

7. Solve $\frac{d^2x}{d^2t}+9x=cos2t$ using L.T. given x (0) =1, x $(\frac{\pi}{2})$ = -1.

Sol: Given $x^{11}+9x=cos2t$

L $[x^{11}]+9[x]=L[cos2t]$

$\Rightarrow\ s^2L[x]-sx(0)-x^1(0)+9L[x\}=\frac{s}{s^2+4}$

$\Rightarrow(s^2+9)L[x]-s-a=\frac{s}{s^2+4}$

$\Rightarrow(s^2+9)L[x]=\frac{s}{s^2+4}+(s+a)$

$L[x]=\frac{s}{(s^2+4((s^2+9)}+\frac{s}{(s^2+9)}+\frac{a}{(s^2+9)}$

X$=L^{-1}[\frac{s}{(s^2+4((s^2+9)}]+L^{-1}[\frac{s}{(s^2+9)}]+L^{-1}[\frac{a}{(s^2+9)}]$

$=\frac{1}{5}L^{-1}[\frac{s}{s^2+4}-\frac{s}{s^2+9}]+cos3t+\frac{a}{3}sin3t$

$=\frac{1}{5}L^{-1}[\frac{s}{s^2+4}]-\frac{1}{5}L^{-1}[\frac{s}{s^2+9}]+cos3t+\frac{a}{3}sin3t$

$=\frac{1}{5}cos2t-\frac{1}{5}cos3t+cos3t+\frac{a}{3}sin3t$ ------------------→ (1)

Given x $(\frac{\pi}{2})$ = -1.

$\therefore\ -1=\frac{1}{5}cos2\left(\frac{\pi}{2}\right)-\frac{1}{5}cos\frac{3\pi}{2}+cos\frac{3\pi}{2}+cos\frac{3\pi}{2}+\frac{a}{3}sin\frac{3\pi}{2}$

$\Rightarrow\quad -1=-\frac{1}{5}-0+0-\frac{a}{3}$

$\frac{a}{3}=-\frac{1}{5}+1$

$\frac{a}{3}=\frac{4}{5}$

$\therefore\quad x=\frac{1}{5}cos2t+\frac{4}{5}cos3t+\frac{4}{5}sin3t$ From (1)

8. $\boldsymbol{Solve(D^3 - 3D^2 + 3D - 1)y = t^2e^t}$ **Using L.T given y (0) =1,** $\boldsymbol{y^1 = 0, y^{11}(0) = -2}$

Sol: Given $y^{111} - 3y^{11} + 3y^1 - y = t^2e^t$

$$L[y^{111}] - 3L[y^{11}] + 3L[y^1] - L[y] = L[t^2e^t]$$

$$\Rightarrow \{s^3L[y] - s^2y(0) - sy^1(0) - y^{11}(0)\} - 3\{s^2L[y] - sy^1(0) - y(0)\} + 3\{sL[y] - y(0)\} - L[y] = L[t^2e^t]$$

$$\Rightarrow (s^3 - 3s^2 + 3s - 1)L[y] - s^2 - 0 + 2 + 0 + 3(1) - 3(1) = (-1)^2\frac{d^2}{ds^2}L[e^t]$$

$$\Rightarrow (s-1)^3\text{L[y]-}s^2 + 2 = \frac{d^2}{ds^2}\left(\frac{1}{s-1}\right)$$

$$= \frac{2}{(s-1)^3}$$

$$\Rightarrow (s-1)^3\text{L[y]} = \frac{2}{(s-1)^3} + s^2 - 2$$

$$L[y] = \frac{2}{(s-1)^6} + \frac{s^2}{(s-1)^3} - \frac{2}{(s-1)^3}$$

$$y = L^{-1}[\frac{2}{(s-1)^6}] + L^{-1}[\frac{s^2}{(s-1)^3}] - L^{-1}[\frac{2}{(s-1)^3}]$$

$$= 2L^{-1}[\frac{1}{(s-1)^6}] + L^{-1}[\frac{s^2}{(s-1)^3}] - 2L^{-1}[\frac{1}{(s-1)^3}]$$

$$=2e^tL^{-1}\left[\frac{1}{(s)^6}\right] + L^{-1}\frac{s^2}{(s-1)^3} - 2e^tL^{-1}\left[\frac{1}{s^3}\right]$$

$$= 2e^t\frac{t^5}{5!} - 2e^t\frac{t^2}{2!} + L^{-1}\left[\frac{s^2}{(s-1)^3}\right]$$

Consider $L^{-1}\left[\frac{s^2}{(s-1)^3}\right]$

W.K.T $L^{-1}\left[\frac{1}{(s-1)^3}\right] = e^tL^{-1}\left[\frac{1}{s^3}\right] = e^t\frac{t^2}{2!} = \frac{e^tt^2}{2}$

$$L^{-1}\left[\frac{s^2}{(s-1)^3}\right] = \frac{d^2}{ds^2}\left(\frac{e^tt^2}{2}\right) = \frac{1}{2}\frac{d}{dt}(2te^t + t^2e^t) = \frac{1}{2}(2e^t + 2te^t + 2te^t + t^2e^t)$$

$$= \frac{1}{2}(2e^t + 4te^t + t^2e^t)$$

$$\therefore \quad y = 2e^t\frac{t^5}{5!} - 2e^t\frac{t^2}{2!} - \frac{1}{2}(2e^t + 4te^t + t^2e^t)$$